**MALLA REDDY ENGINEERING COLLEGE (AUTONOMOUS)**
**DEPARTMENT OF MECHANICAL ENGINEERING**

**II B.TECH II nd SEM MECHANICAL ENGINEERING**

**Name of the Subject : Dynamics of Machinery**

**Name of the Faculty : Mr.E.Venkata Reddy/ MrS.Uday Kumar**

## SYLLABUS

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**Course Code: 50312** **Credits: 4**

**B.Tech. – IV Semester**

**DYNAMICS OF MACHINES**

**Pre-requisite:** Engineering Mechanics and Kinematics of Machinery

**Objectives:** The objective of this subject is to know static and dynamic behavior of mechanisms under different loading conditions.

**MODULE – I: Precession** **[8 Periods]**

Precession: Gyroscopes, effect of precession motion on the stability of moving vehicles such as motor car, motor cycle, aero planes and ships.

**MODULE–II: Static and Dynamic Force Analysis of Planar Mechanisms & Synthesis of Linkages** **[14 Periods]**

A: Static And Dynamic Force Analysis Of Planar Mechanisms: Introduction -Free Body Diagrams – Conditions for equilibrium – Two, Three and Four Members – Inertia forces and D'Alembert's Principle – planar rotation about a fixed centre.

B: Synthesis Of Linkages: Three position synthesis – Four position Synthesis – Precision positions – Structural error – Chebychev's spacing, Freudentein's equation, Problems.

**MODULE - III: Clutches & Turning Moment Diagram and Fly Wheels** **[14 Periods]**

A: Clutches: Friction clutches- Single Disc or plate clutch, Multiple Disc Clutch, Cone Clutch, Centrifugal Clutch. Brakes and Dynamometers: Simple block brakes, internal expanding brake, band brake of vehicle. Dynamometers – absorption and transmission types. General description and methods of operations.

B: Turning Moment Diagram and Fly Wheels: Turning moment – Inertia Torque connecting rod angular velocity and acceleration, crank effort and torque diagrams – Fluctuation of energy – Fly wheels and their design.

**MODULE - IV: Balancing & Vibration** **[14 Periods]**

A: Balancing: Balancing of rotating masses Single and multiple – single and different planes. Balancing of Reciprocating Masses- Primary, Secondary, and higher balancing of reciprocating masses. Analytical and graphical methods.Unbalanced forces and couples – examination of –'V' multi cylinder in line and radial engines for primary and secondary balancing, locomotive balancing.

B: Vibration: Free Vibration of mass attached to vertical spring – Simple problems on forced damped vibration, Vibration Isolation & Transmissibility Whirling of shafts, critical speeds, torsional vibrations, two and three rotor systems.

**MODULE - V: Governers** **[10 Periods]**

Governers: Watt, Porter and Proell governors. Spring loaded governors – Hartnell and hartung with auxili ary springs. Sensitiveness, isochronism and hunting.

**TEXT BOOKS :**

1. Theory of Machines / S.S Ratan/ Mc. Graw Hill Publ.
2. Theory of Machines / Jagadish Lal & J.M.Shah / Metropolitan.

**REFERENCES:**

1. Mechanism and Machine Theory / JS Rao and RV Dukkipati / New Age

2. Theory of Machines / Shiegly / MGH
3. Theory of Machines / Thomas Bevan / CBS Publishers
4. Theory of machines / Khurmi/S.Chand.

COURSE OUTCOME:

1. After completion of the course, students will be able to:
2. Understand the concept of gyroscope and understand and analyze the effect of precision on different types of vehicles.
3. Learn the concept of free body diagram, preparing of free body diagram and can do the analysis of members which are subjected to different types of forces and do the synthesis of linkages.
4. Learn the concepts of clutches, brakes and dynamometers and able to analyze various types of clutches, brakes, dynamometers and learn the concept of turning moment diagram and its analysis for various types of engines and design the flywheels.
5. Know various types of forces that are acting on the rotating masses and necessity of balancing and balancing of various types of engines and their analysis and learn the concept of vibrations and get depth knowledge about different types of vibrations.
6. Learn the concept of governors and analyze various types of governors and get familiar with various terms associated with governors.

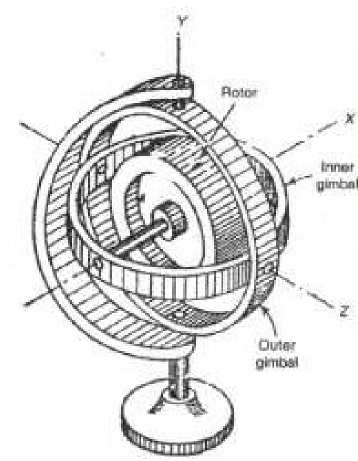
**MODULE-1**

## PRECESSION

Introduction

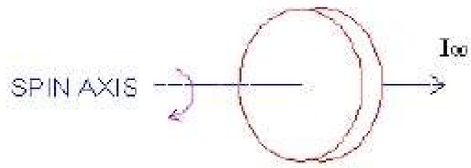
*'Gyre'* is a Greek word, meaning 'circular motion' and Gyration means the whirling motion. A gyroscope is a spatial mechanism which is generally employed for the study of precessional motion of a rotary body. Gyroscope finds applications in gyrocompass, used in aircraft, naval ship, control system of missiles and space shuttle. The gyroscopic effect is also felt on the automotive vehicles while negotiating a turn.

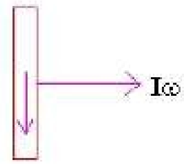
A gyroscope consists of a rotor mounted in the inner gimbal. The inner gimbal is mounted in the outer gimbal which itself is mounted on a fixed frame as shown in Fig.. When the rotor spins about X-axis with angular velocity ω rad/s and the inner gimbal precesses (rotates) about Y-axis, the spatial mechanism is forced to turn about Z-axis other than its own axis of rotation, and the gyroscopic effect is thus setup. The resistance to this motion is called gyroscopic effect.




## ANGULAR MOTION

A rigid body, (Fig.) spinning at a constant angular velocity ω rad/s about a spin axis through the mass centre. The angular momentum 'H' of the spinning body is represented by a **vector** whose magnitude is 'Iω'. I represents the mass amount of inertia of the rotor about the axis of spin.

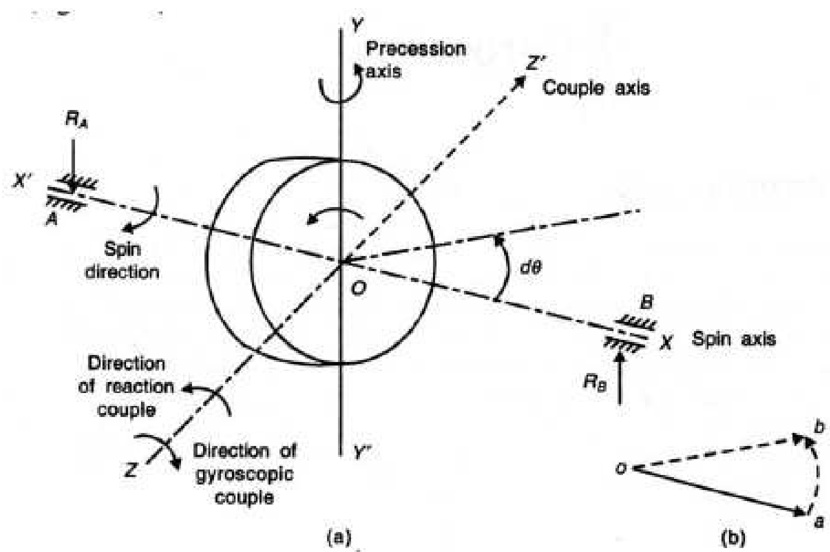



The direction of the angular momentum can be found from the right hand screw rule or the right hand thumb rule. Accordingly, if the fingers of the right hand are bent in the direction of rotation of rotor, then the thumb indicates the direction of momentum.

## GYROSCOPIC COUPLE

Consider a rotary body of mass *m* having radius of gyration *k* mounted on the shaft supported at two bearings. Let the rotor spins (rotates) about X-axis with constant angular velocity ω rad/s. The X-axis is, therefore, called spin axis, Y-axis, precession axis and Z-axis, the couple or torque axis (Fig.).




The angular momentum of the rotating mass is given by,

$$H = mk^2 \omega = I\omega$$

Now, suppose the shaft axis (X-axis) precesses through a small angle δθ about Y-axis in the plane *XOZ,* then the angular momentum varies from *H* to *H* + δ*H,* where δ*H* is the change in the angular momentum, represented by vector ab [Figure 15.2(b)]. For the small value of angle of rotation 5θ, we can write

$$ab = oa \times \delta\theta$$
$$\delta H = H \times \delta\theta$$
$$= I\omega\delta\theta$$

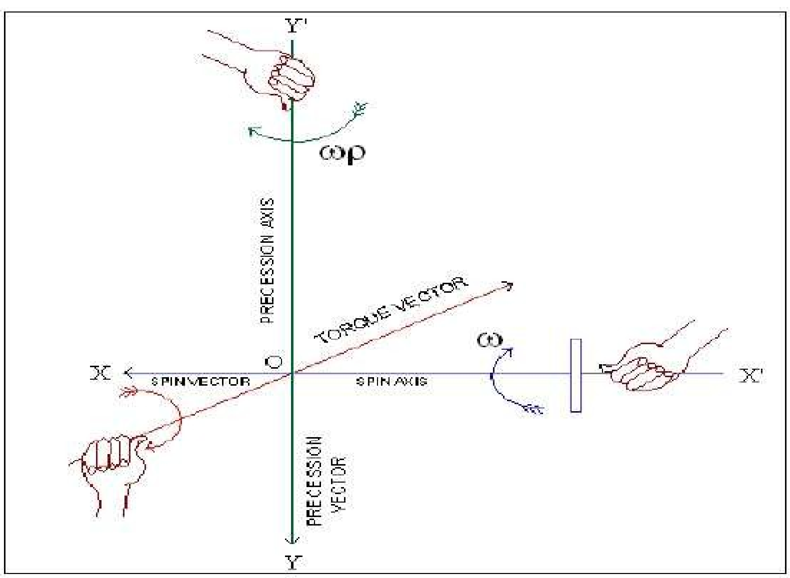
However, the rate of change of angular momentum is:

$$C = \frac{dH}{dt} = \lim_{\delta t \to 0}\left(\frac{I\omega\,\delta\theta}{\delta t}\right)$$
$$= I\omega\frac{d\theta}{dt}$$

$$\mathbf{C = I\omega\omega_p}$$

## Direction of Spin vector, Precession vector and Couple/Torque vector with forced precession

To determine the direction of spin, precession and torque/couple vector, right hand screw rule or right hand rule is used. The fingers represent the rotation of the disc and the thumb shows the direction of the spin, precession and torque vector (Fig.).

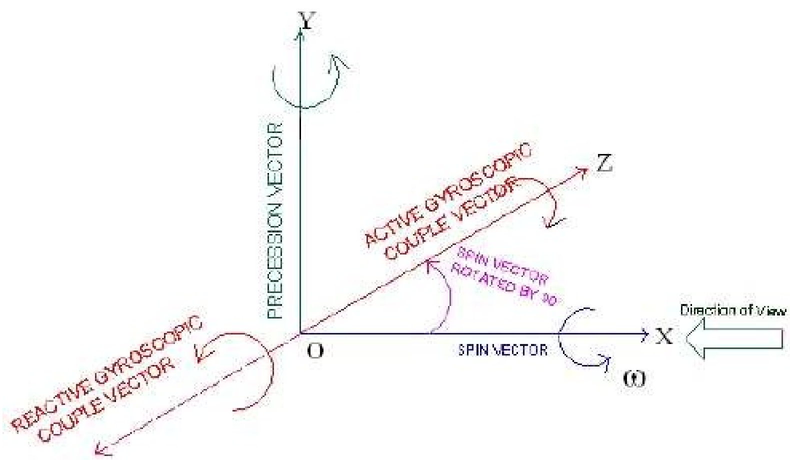


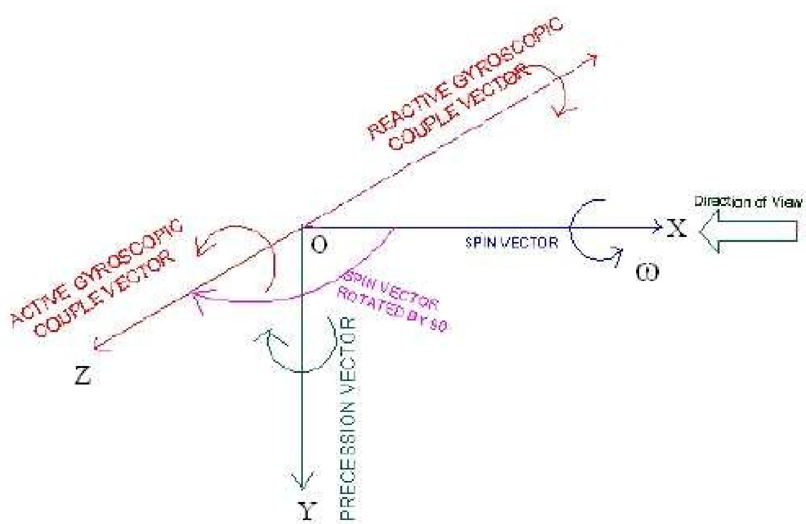
The method of determining the direction of couple/torque vector is as follows

**Case (i):**

Consider a rotor rotating in anticlockwise direction when seen from the right (Fig.5 and Fig. 6), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector, the following procedure is used.

- Turn the spin vector through $90^0$ in the direction of precession on the XOZ plane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

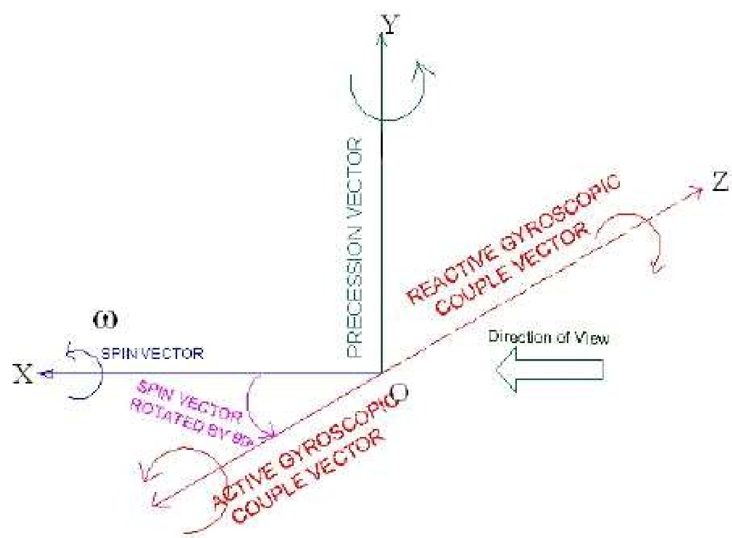


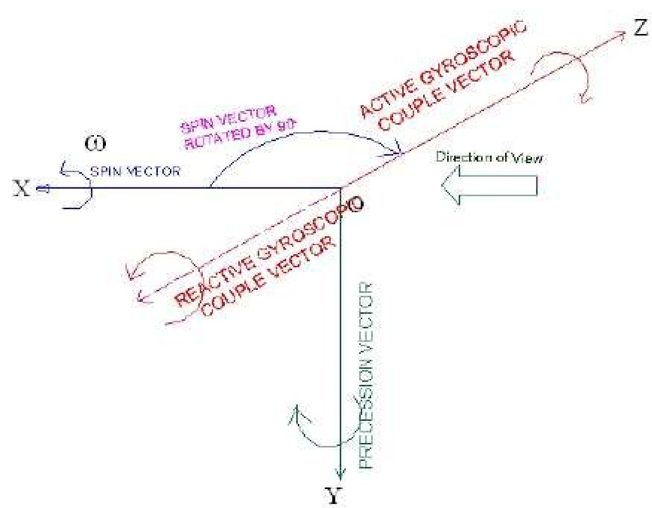



**Case (ii):**

Consider a rotor rotating in clockwise direction when seen from the right (Fig.7 and Fig. 8), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector,

- Turn the spin vector through $90^0$ in the direction of precession on the XOZ plane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torque vector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction



The resisting couple/ reactive couple will act in the direction opposite to that of the gyroscopic couple. This means that, whenever the axis of spin changes its direction, a **gyroscopic couple** is applied to it through the bearing which supports the spinning axis.

## GYROSCOPIC EFFECT ON SHIP

Gyroscope is used for stabilization and directional control of a ship sailing in the rough sea. A ship, while navigating in the rough sea, may experience the following three different types of motion:

(i) Steering—The turning of ship in a curve while moving forward

(ii) Pitching—The movement of the ship up and down from horizontal position in a vertical plane about transverse axis

(iii)Rolling—Sideway motion of the ship about longitudinal axis

For stabilization of a ship against any of the above motion, the major requirement is that the gyroscope shall be made to precess in such a way that reaction couple exerted by the rotor opposes the disturbing couple which may act on the frame.

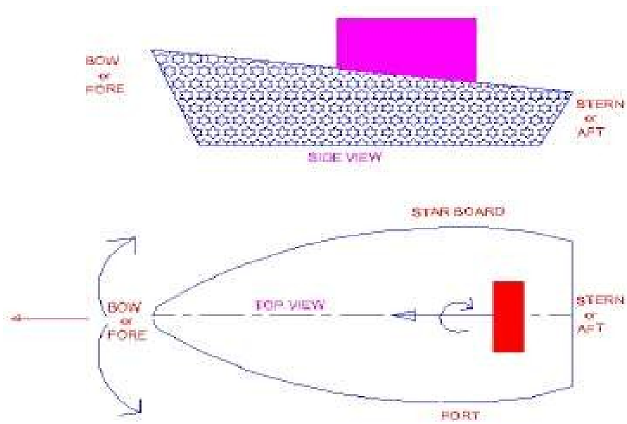
**Ship Terminology**

(i) Bow – It is the fore end of ship

(ii) Stern – It is the rear end of ship

(iii) Starboard – It is the right hand side of the ship looking in the direction of motion

(iv) Port – It is the left hand side of the ship looking in the direction of motion

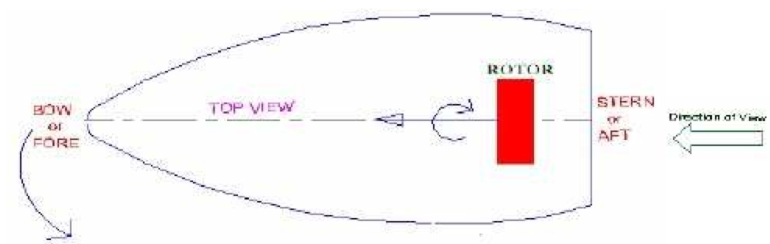


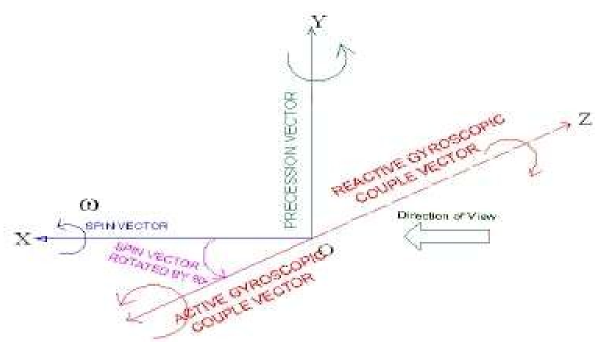
Consider a gyro-rotor mounted on the ship along longitudinal axis (X-axis) as shown in Fig.10 and rotate in clockwise direction when viewed from rear end of the ship. The angular speed of the rotor is ω rad/s. The direction of angular momentum vector *oa*, based on direction of rotation of rotor, is decided using right hand thumb rule as discussed earlier. The gyroscopic effect during the three types of motion of ship is discussed.

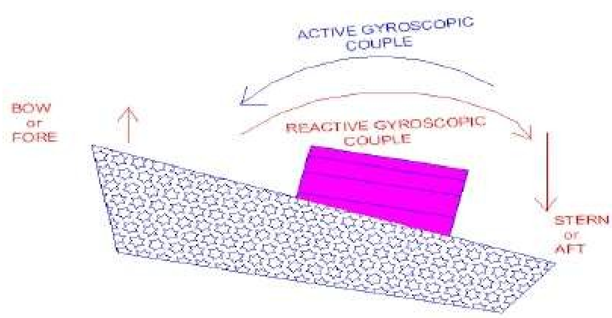
**Gyroscopic effect on Steering of ship**

***(i) Left turn with clockwise rotor***

When ship takes a left turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way.



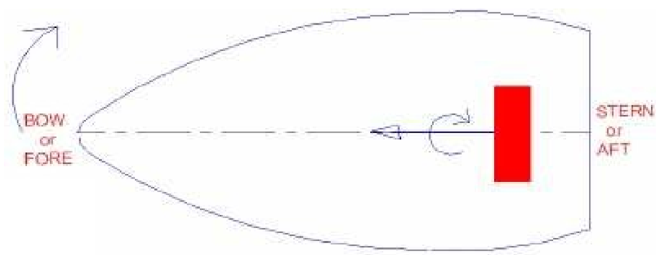



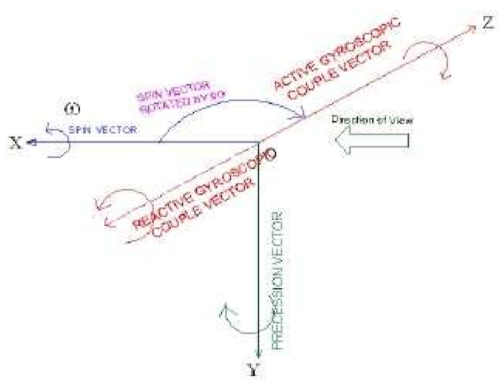



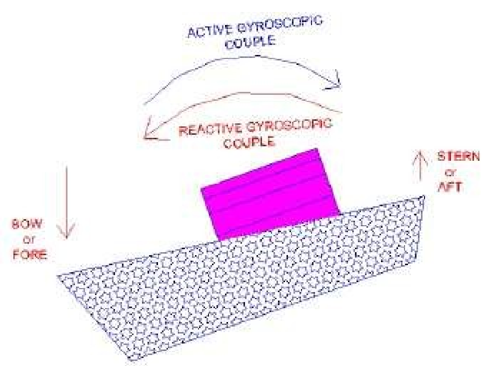
Note that, always reactive gyroscopic couple is considered for analysis. From the above analysis (Fig.), the couple acts over the ship between stern and bow. This reaction couple tends to raise the front end (bow) and lower the rear end (stern) of the ship.

***(ii) Right turn with clockwise rotor***

When ship takes a right turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple acts on the ship is analyzed (Fig 14). Again, the couple acts in vertical plane, means between stern and bow. Now the reaction couple tends to lower the bow of the ship and raise the stern.

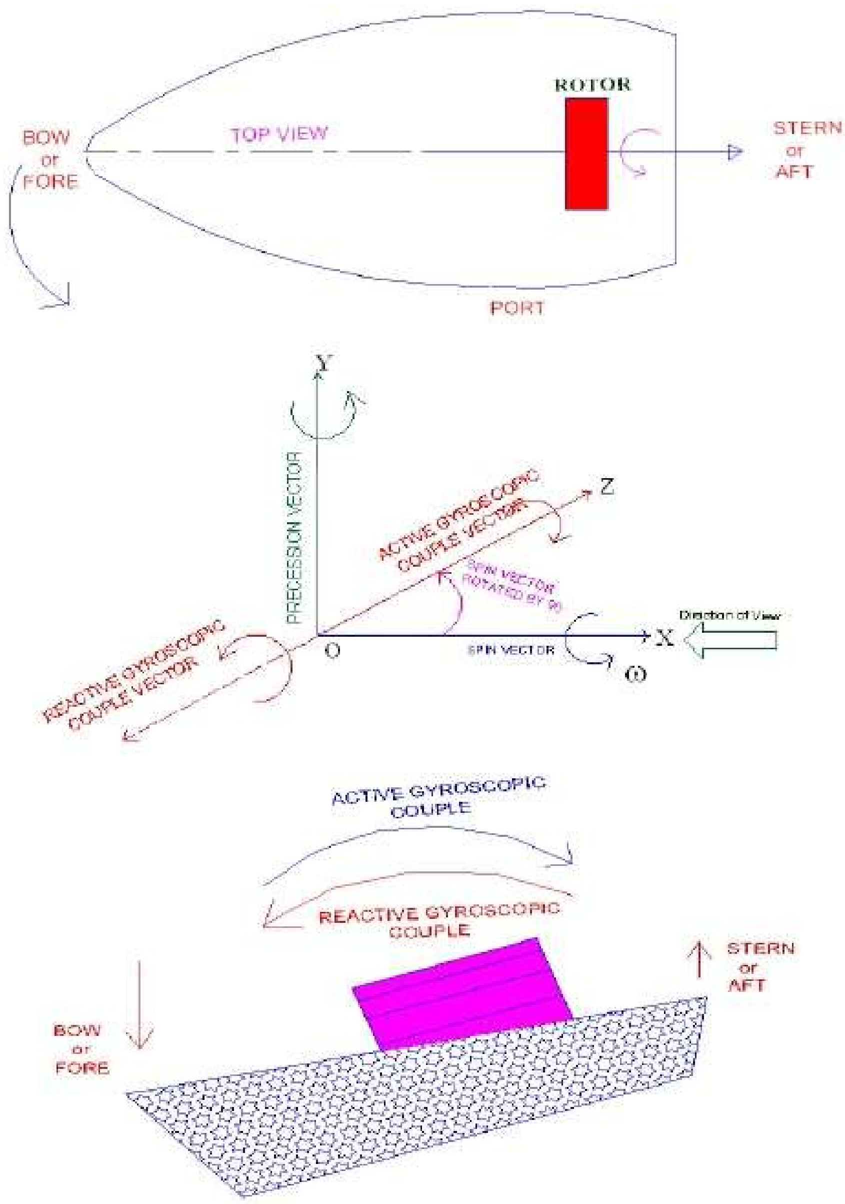


***(iii) Left turn with anticlockwise rotor***

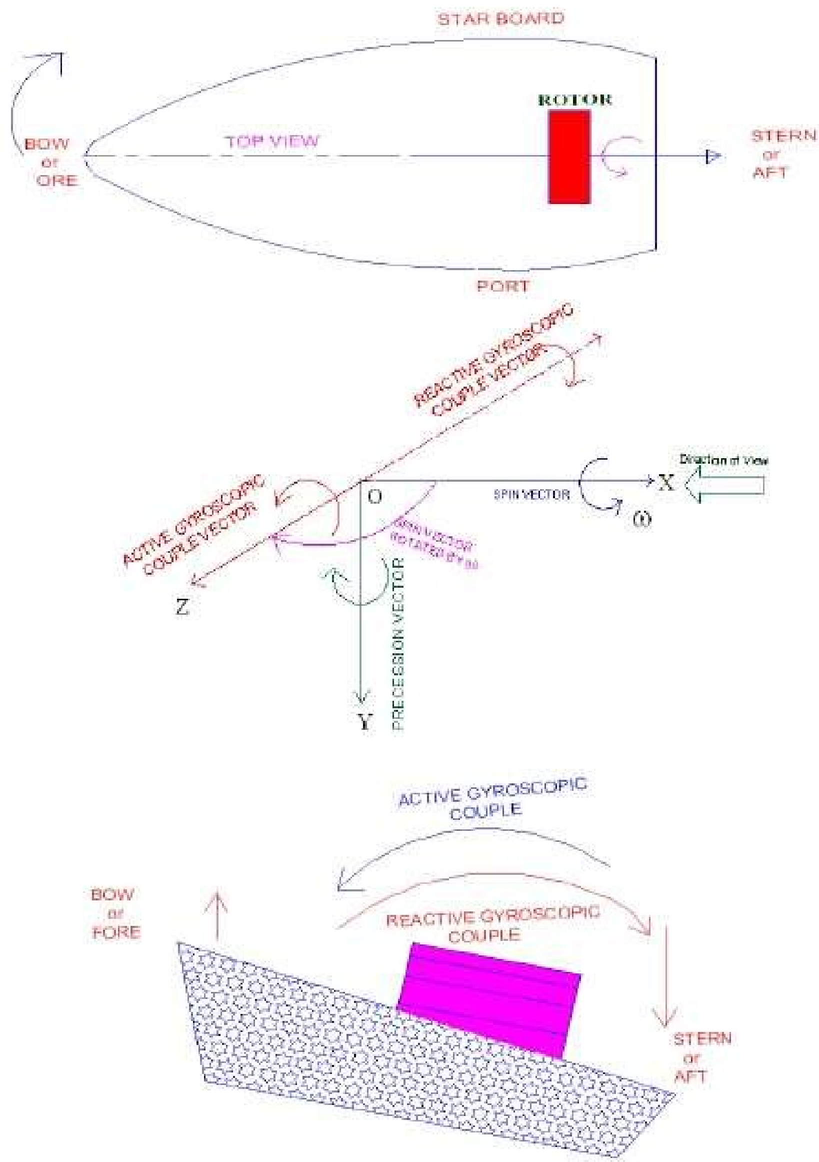
When ship takes a left turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way (Fig.).




The couple acts over the ship is between stern and bow. This reaction couple tends to press or dip the front end (bow) and raise the rear end (stern) of the ship.

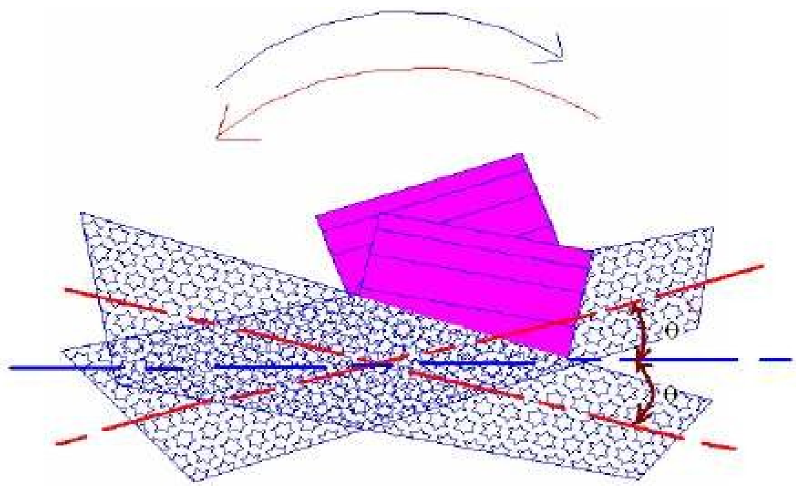
***(iv) Right turn with anticlockwise rotor***

When ship takes a right turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is according to Fig 20. Now, the reaction couple tends to raise the bow of the ship and dip the stern



**Gyroscopic effect on Pitching of ship**

The pitching motion of a ship generally occurs due to waves which can be approximated as sine wave. During pitching, the ship moves up and down from the horizontal position in vertical plane (Fig. )




Let $\theta$ = angular displacement of spin axis from its mean equilibrium position
$A$ = amplitude of swing
(= angle in degree $\times \frac{2\pi}{360^\circ}$)

and $\omega_0$ = angular velocity of simple hormonic motion $\left(= \frac{2\pi}{\text{time period}}\right)$

The angular motion of the rotor is given as

$$\theta = A \sin \omega_0 t$$

Angular velocity of precess: $\omega_p = \frac{d\theta}{dt}$

$$= \frac{d}{dt}(A \sin \omega_0 t)$$

or $\omega_p = A\omega_0 \cos \omega_0 t$

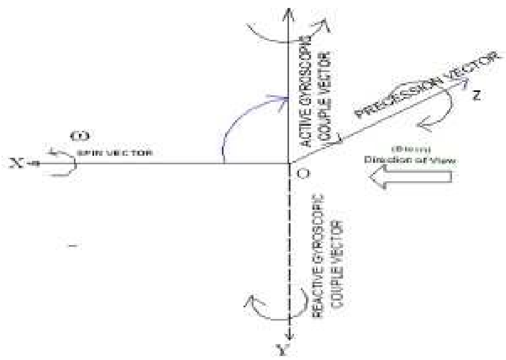
The angular velocity of precess will be maximum when $\cos \omega_0 t = 1$

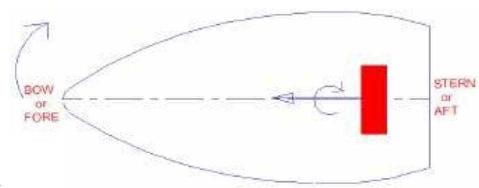
or $\omega_{p\max} = A\omega_0$

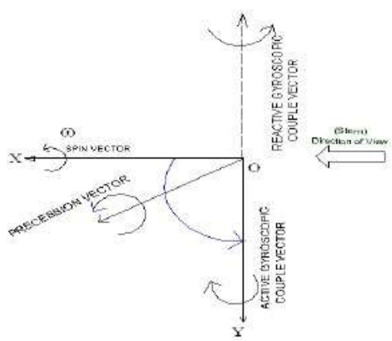
$$= A \times \frac{2\pi}{t}$$

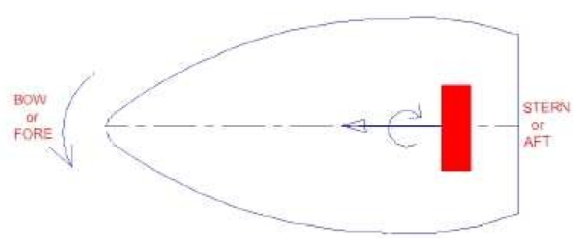
Thus the gyroscopic couple: $C = I\,\omega\,\omega_p$

Consider a rotor mounted along the longitudinal axis and rotates in clockwise direction when seen from the rear end of the ship. The direction of momentum for this condition is shown by vector *ox* (Fig.24). When the ship moves up the horizontal position in vertical plane by an angle δθ from the axis of spin, the rotor axis (X-axis) processes about Z- axis in XY-plane and for this case Z-axis becomes precession axis. The gyroscopic couple acts in anticlockwise direction about Y-axis and the reaction couple acts in opposite direction, i.e. in clockwise direction, which tends to move towards **right side** (Fig.25). However, when the ship pitches down the axis of spin, the direction of reaction couple is reversed and the ship turns towards **left side** (Fig.)




Similarly, for the anticlockwise direction of the rotor viewed from the rear end (Stern) of the ship, the analysis may be done.

**Gyroscopic effect on Rolling of ship**.

The axis of the rotor of a ship is mounted along the longitudinal axis of ship and therefore, there is **no** precession of this axis. Thus, no effect of gyroscopic couple on the ship frame is formed when the ship rolls

## Gyroscopic Effect on Aeroplane

Aeroplanes are subjected to gyroscopic effect when it taking off, landing and negotiating left or right turn in the air.

Let

$\omega$ = Angular velocity of the engine rotating parts in rad/s,

m = Mass of the engine and propeller in kg,

$r_W$ = Radius of gyration in m,

I = Mass moment of inertia of engine and propeller in kg $m^2$,

V = Linear velocity of the aeroplane in m/s,

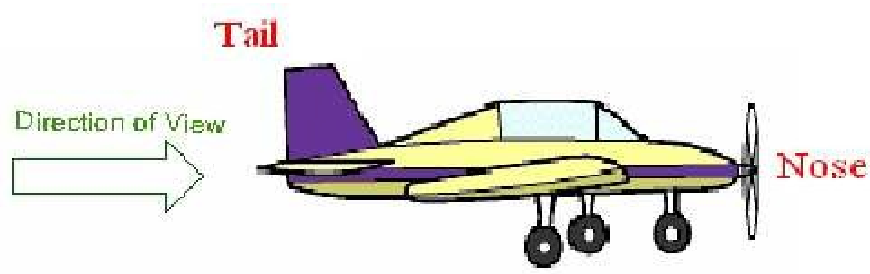
R = Radius of curvature in m,

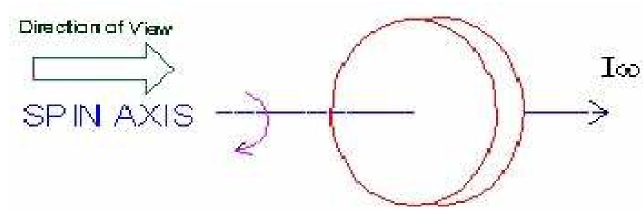
$\omega_p$ =Angular velocity of precession =v/R rad/s

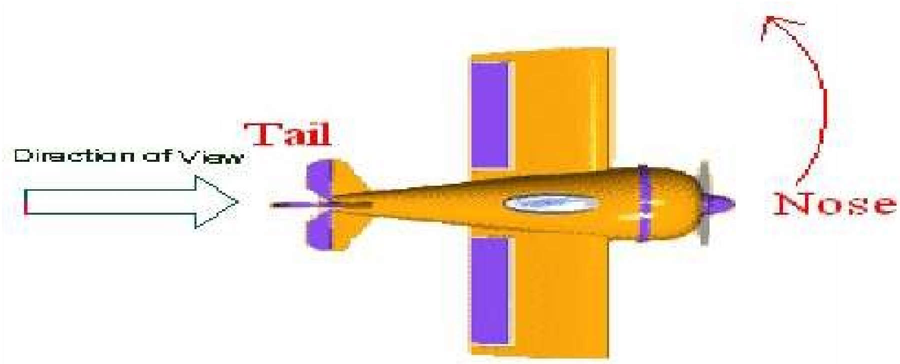
Gyroscopic couple acting on the aero plane $= C = I\,\omega\,\omega_p$

Let us analyze the effect of gyroscopic couple acting on the body of the aero plane for various conditions.

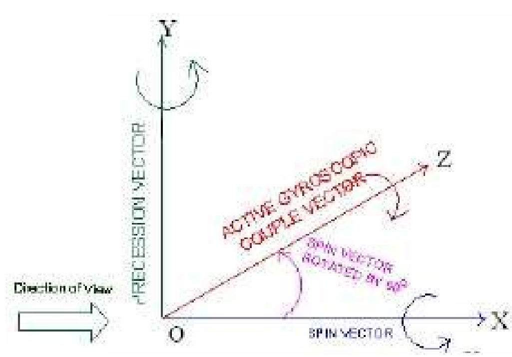
Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

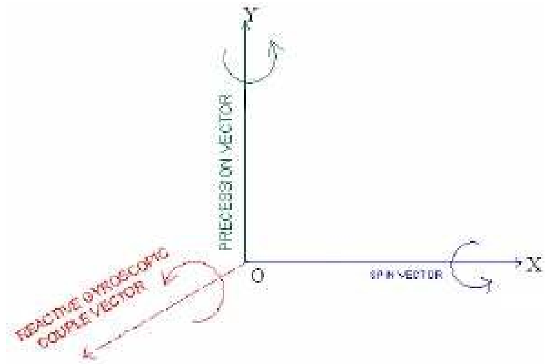



Direction of View
SPIN AXIS
Iω

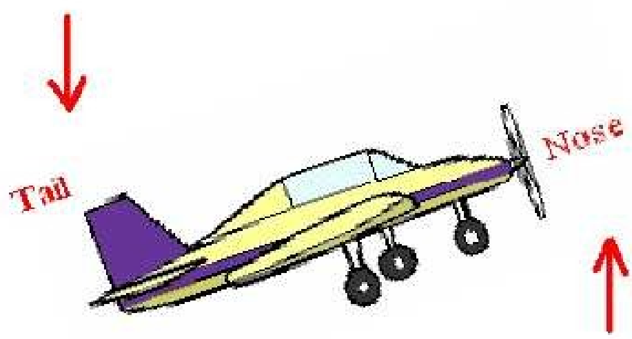

Tail
Direction of View
Nose


Direction of View
O
SPIN VECTOR
X
Iω

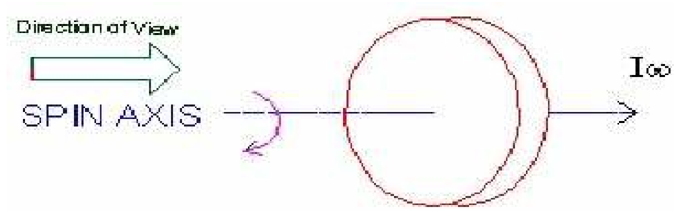

Y
PRECESSION VECTOR
ACTIVE GYROSCOPIC COUPLE VECTOR
Z
Direction of View
O
SPIN VECTOR
X

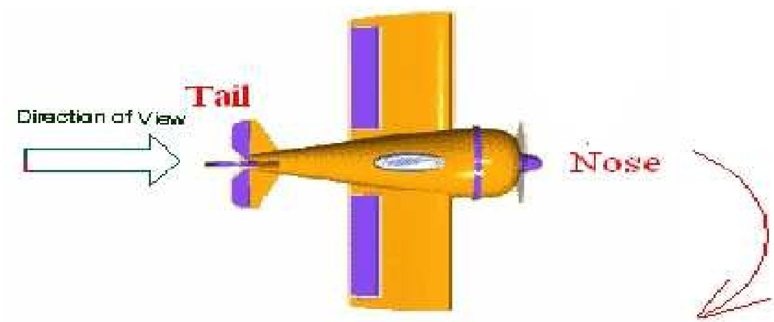



According to the analysis, the reactive gyroscopic couple tends to dip the tail and raise the nose of aeroplane.

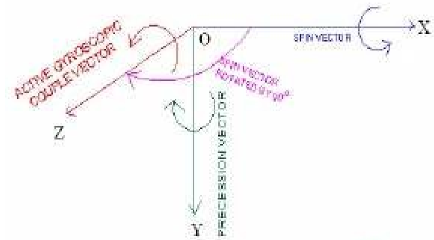


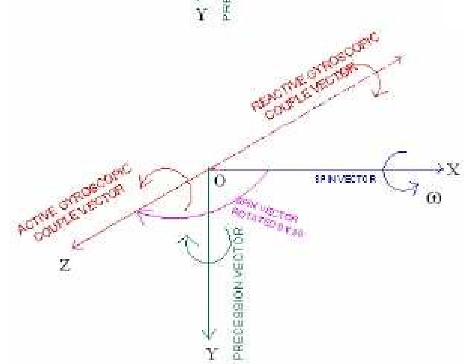

Case (ii): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT


Direction of View
SPIN AXIS
Iω

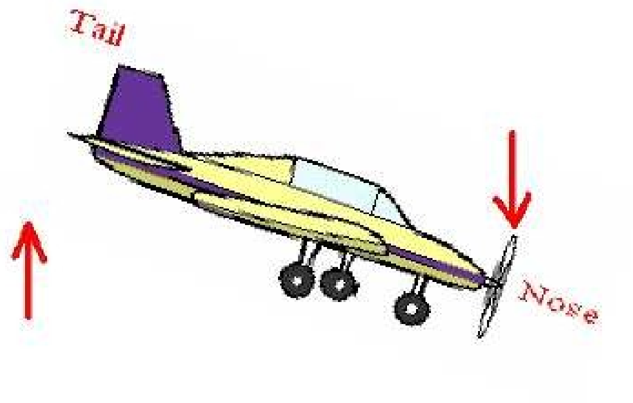

Tail
Direction of View
Nose


Direction of View
O
SPIN VECTOR
X
Iω

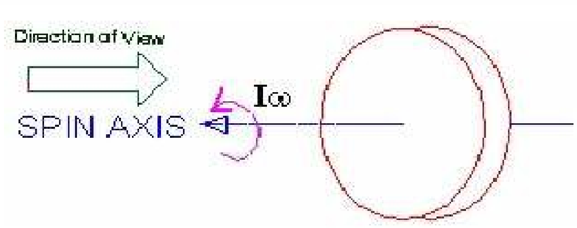

ACTIVE GYROSCOPIC COUPLE VECTOR
O
SPIN VECTOR
X
SPIN VECTOR ROTATED BY 90°
Z
PRECESSION VECTOR
Y

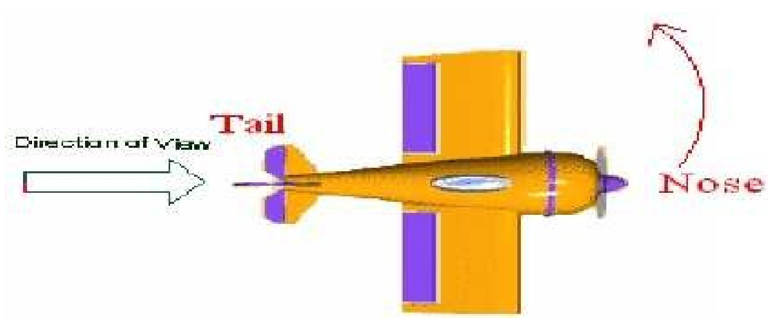

REACTIVE GYROSCOPIC COUPLE VECTOR
ACTIVE GYROSCOPIC COUPLE VECTOR
O
SPIN VECTOR
X
ω
SPIN VECTOR
Z
PRECESSION VECTOR
Y

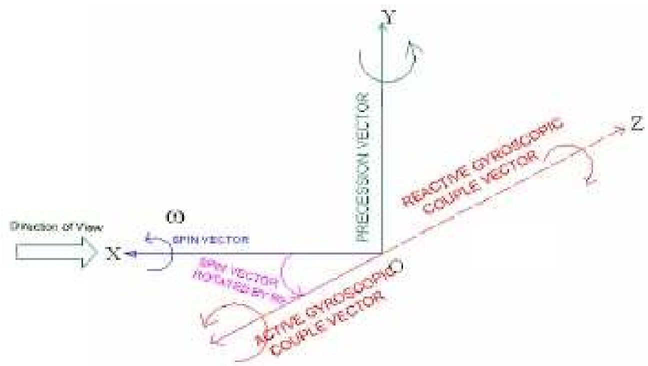
According to the analysis, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.



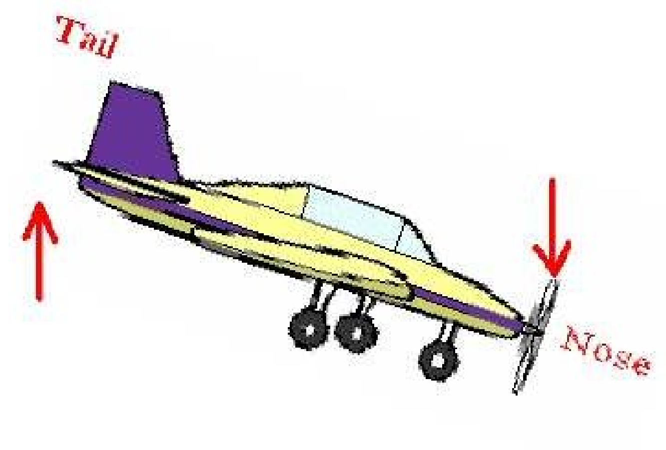

Case (iii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

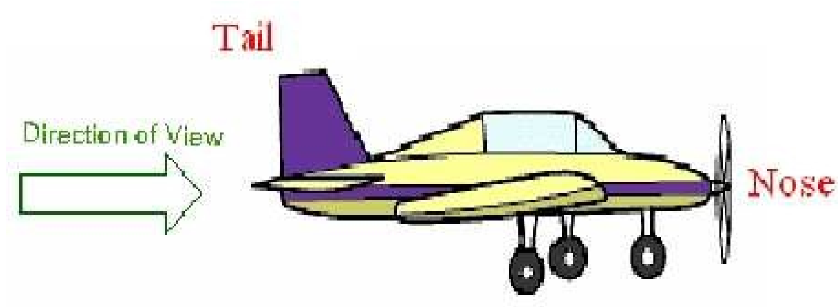



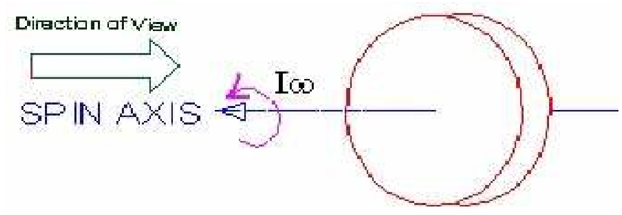


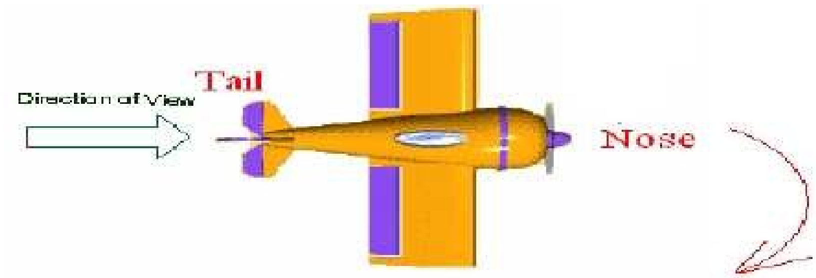

The analysis indicates, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

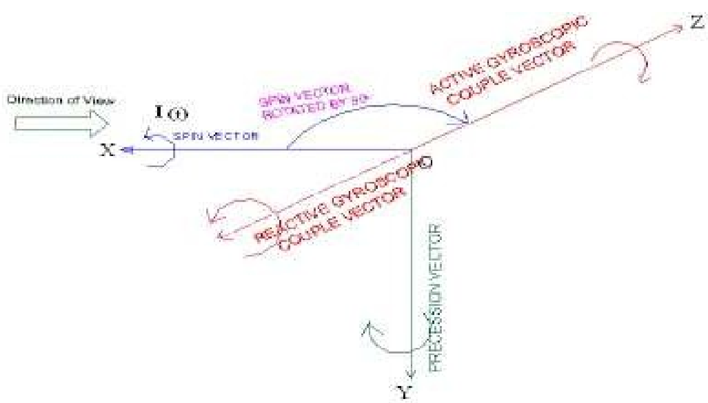



Case (iv): **PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT**



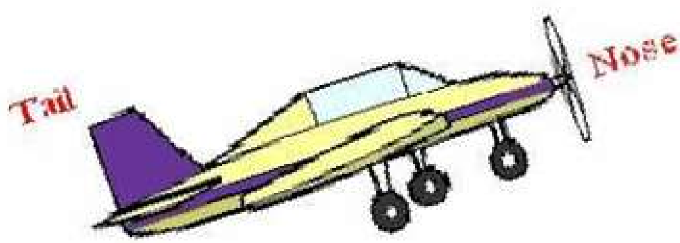


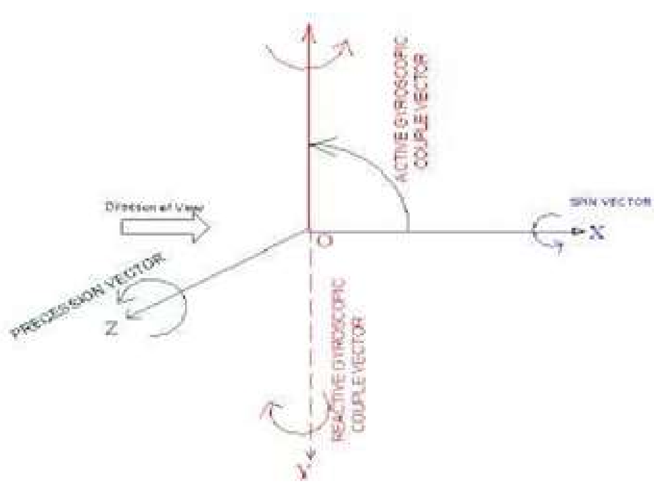



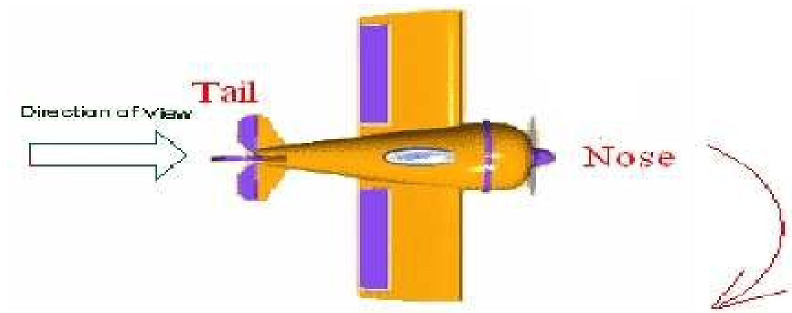

The analysis shows, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.


Case (v): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards

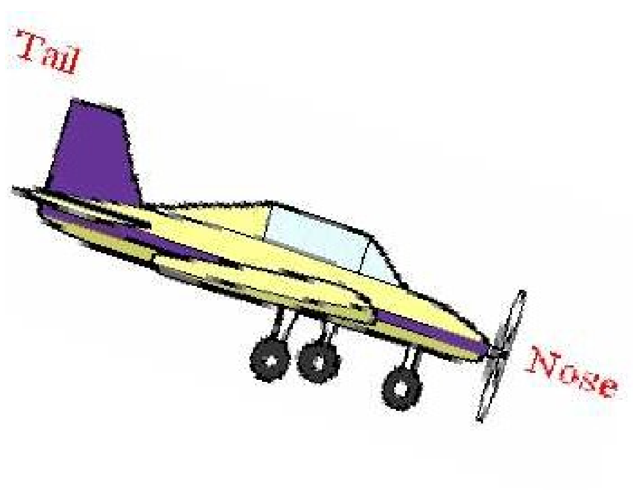


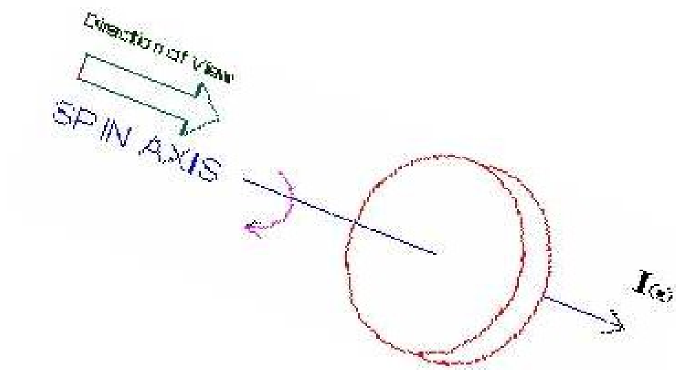



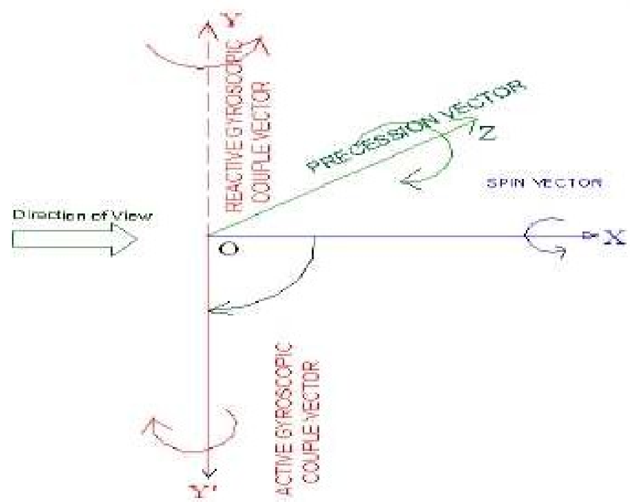
The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right



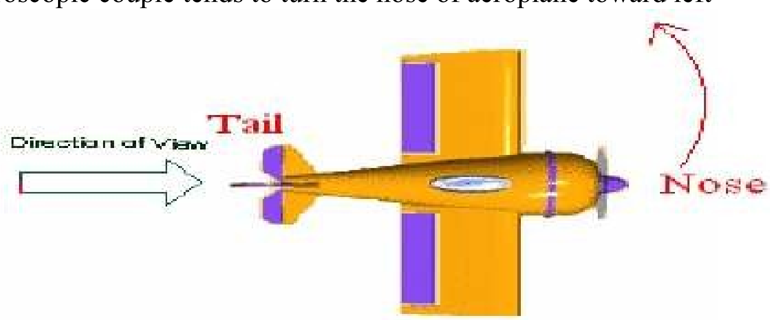

Case (vi): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards



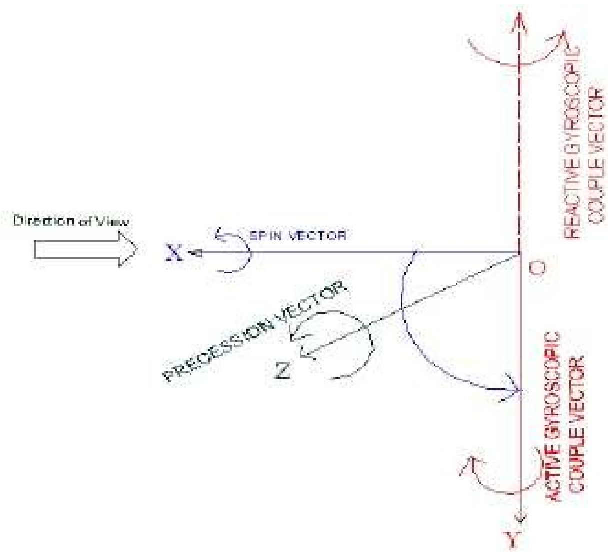


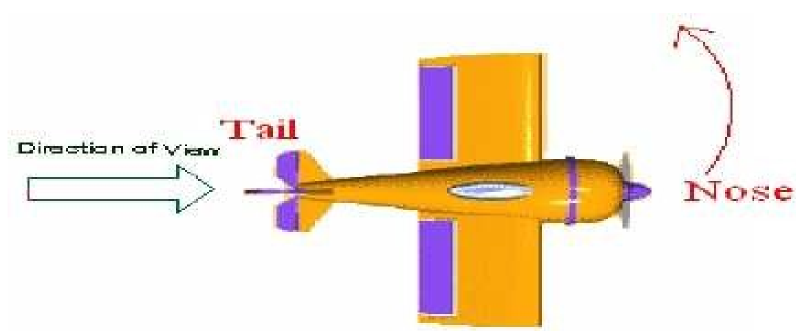

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left



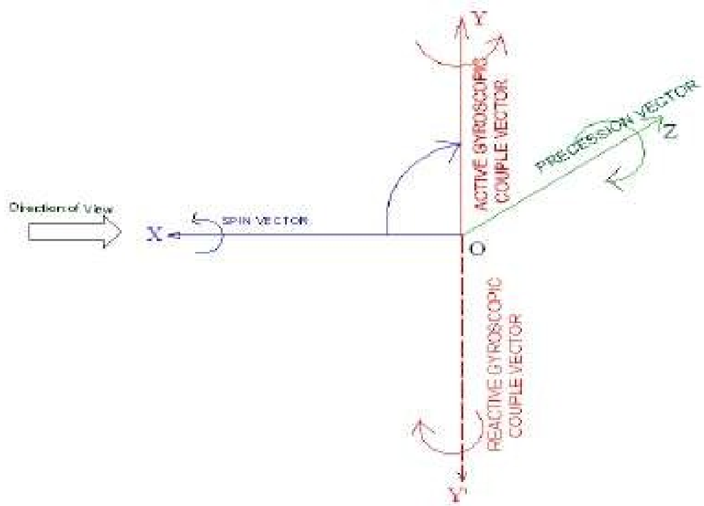
Case (vii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards



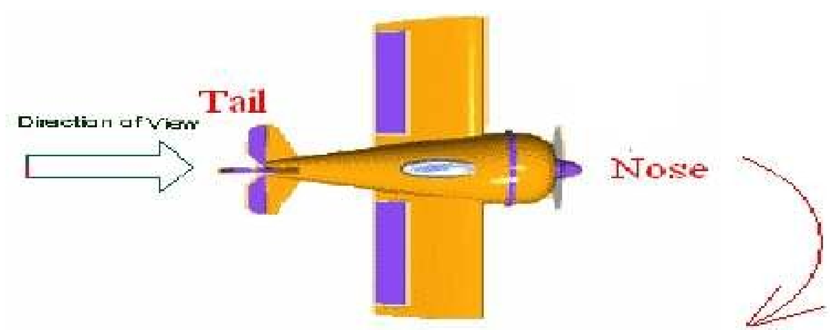

The reactive gyroscopic couple tends to turn the nose of aeroplane toward left



Case (viii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards




The analysis show, the reactive gyroscopic couple tends to turn the nose of aeroplane toward right



## Stability of Automotive Vehicle

A vehicle running on the road is said to be stable when no wheel is supposed to leave the road surface. In other words, the resultant reactions by the road surface on wheels should act in upward direction. For a moving vehicle, one of the reaction is due to gyroscopic couple produced by the rotating wheels and rotating parts of the engine. Let us discuss stability of two and four wheeled vehicles when negotiating a curve/turn.

## Stability of Two Wheeler negotiating a turn

Fig shows a two wheeler vehicle taking **left turn** over a curved path. The vehicle is inclined to the vertical for equilibrium by an angle θ known as angle of heel.

Let

$m$ = *Mass of the vehicle and its* rider in kg,

$W$ = *Weight of the vehicle and* its rider in newtons = $m.g$,

$h$ = *Height of the centre of* gravity of the vehicle and rider,

$r_W$ = *Radius of the wheels,*

$R$ = *Radius of track or* curvature,

$I_W$ = *Mass moment of inertia of each wheel,*

$I_E$ = *Mass moment of inertia of the rotating parts of the engine,*

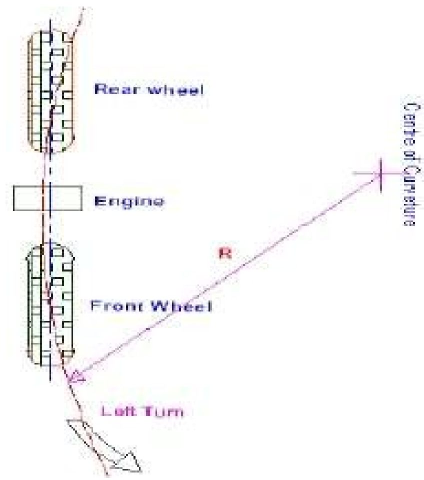
$\omega_W$ = Angular velocity of the wheels,

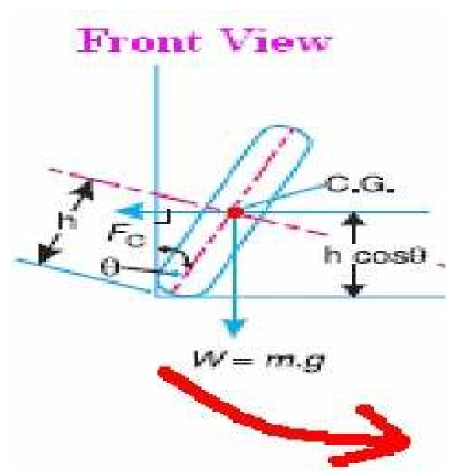
$\omega_E$ = Angular velocity of the engine rotating parts,

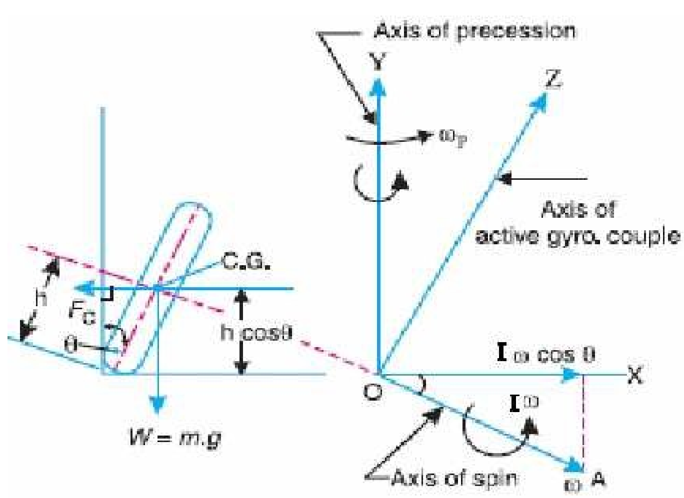
$G$ = *Gear ratio* = $\omega_E / \omega_W$,

*v = Linear velocity of the vehicle =* $\omega_w \times r_w$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium



Let us consider the effect of the gyroscopic couple and centrifugal couple on the wheels.

## 1. Effect of Gyroscopic Couple

We know that, $V = \omega_W \times r_W$

$\omega_E = G . \omega_W$ or

Angular momentum due to wheels = 2 $I_W \omega_W$

Angular momentum due to engine and transmission = $I_E \omega_E$

Total angular momentum ($I x\omega$) = 2 $I_W \omega_W \pm I_E \omega_E$

$$= 2I_w \frac{v}{r_w} \pm I_E G \frac{v}{r_w}$$

$$= \frac{v}{r_w}(2I_w \pm GI_E)$$

Velocity of precession = $\omega_p$

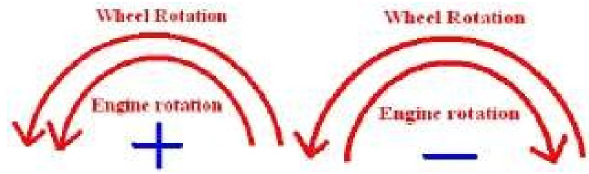
It is observed that, when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig… This angle is known as 'angle of heel'. In other words, the axis of spin is inclined to the horizontal at an angle θ , as shown in Fig.73 Thus, the angular momentum vector I $\omega$ due to spin is represented by OA inclined to OX at an angle θ. But, the precession axis is in vertical. Therefore, the spin vector is resolved along OX.

Gyroscopic Couple,

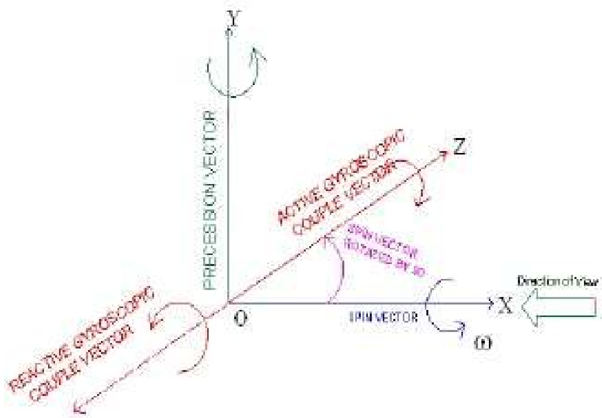
$$C_g = (I\omega)\cos\theta \times \omega_p$$

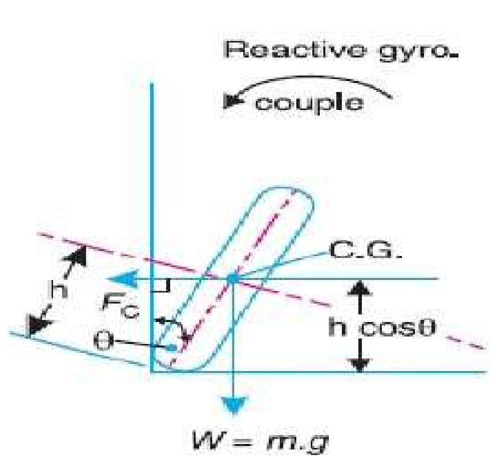
$$C_g = \frac{v^2}{Rr_w}(2I_w \pm GI_E)\cos\theta$$

Note: When the engine is rotating in the same direction as that of wheels, then the positive sign is used in the above equation. However, if the engine rotates in opposite direction to wheels, then negative sign is used.

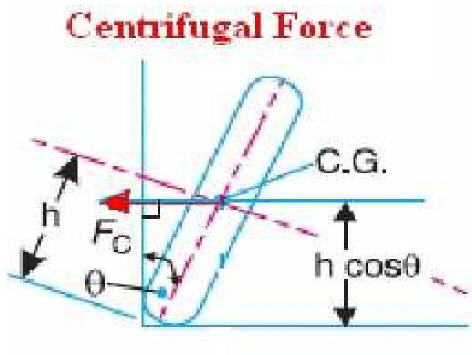



The gyroscopic couple will act over the vehicle outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig…



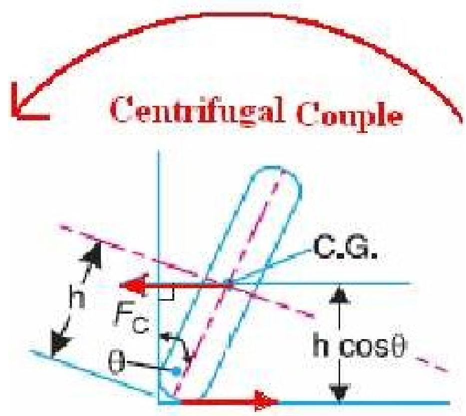
## 2. Effect of Centrifugal Couple




Centrifugal force, $F_c = \frac{mv^2}{R}$

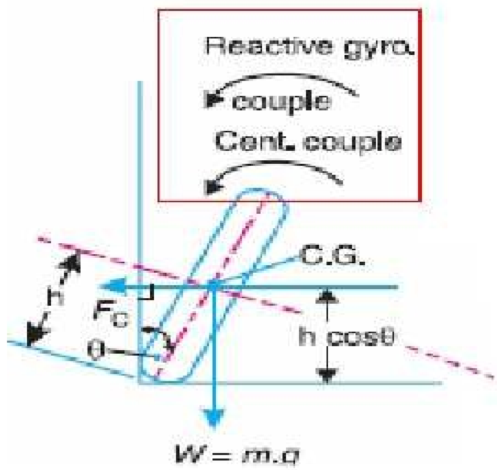
Centrifugal Couple

$$C_c = F_c \times h\cos\theta$$
$$= \frac{mv^2}{R} h\cos\theta$$




The Centrifugal couple will act over the two wheeler outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig.

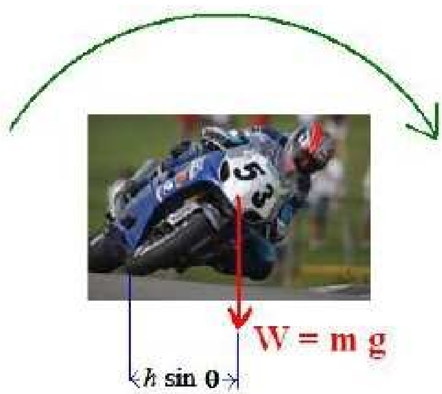
Therefore, the total Over turning couple: $C = C_g + C_c$




$$C = \frac{v^2}{Rr}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta$$

For the vehicle to be in equilibrium, overturning couple should be equal to balancing couple acting in clockwise direction due to the weight of the vehicle and rider.

$$C = mgh \sin\theta$$




For the stability, overturning couple must be equal to balancing couple,

$$\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R} h\cos\theta = mgh\sin\theta$$

Therefore, from the above equation, the value of angle of heel (θ) may be determined, so that the vehicle does not skid. Also, for the given value of θ, the maximum vehicle speed in the turn with out skid may be determined.

## Stability of Four Wheeled Vehicle negotiating a turn.

Consider a four wheels automotive vehicle as shown in Figure 82. The engine is mounted at the rear with its crank shaft parallel to the rear axle. The centre of gravity of the vehicle lies vertically above the ground where total weight of the vehicle is assumed to be acted upon.

Let

$m$ = *Mass of the vehicle* (kg)

$W$ = *Weight of the vehicle* (N) = $m.g$,

$h$ = *Height of the centre of* gravity of the vehicle (m)

$r_W$ = *Radius of the wheels (m)*

$R$ = *Radius of track or* curvature (m)

$I_W$ = *Mass moment of inertia of each wheel (kg-m*$_2$*)*

$I_E$ = *Mass moment of inertia of the rotating parts of the engine (kg-m*$_2$*)*

$\omega_W$ = Angular velocity of the wheels (rad/s)

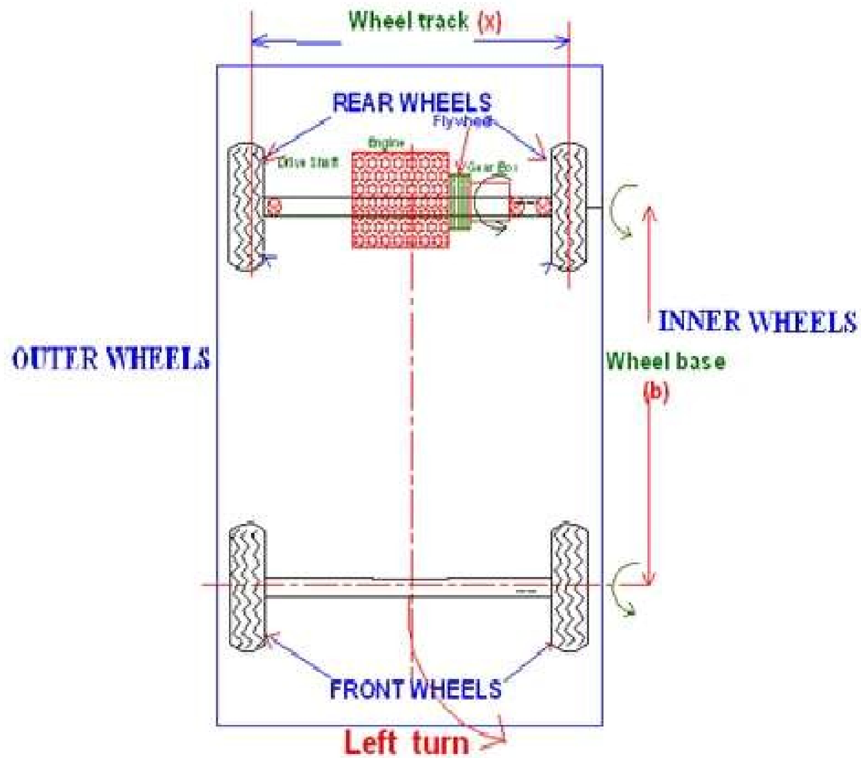
$\omega_E$ = Angular velocity of the engine (rad/s)

$G$ = *Gear ratio* = $\omega_E / \omega_W$,

$v$ = *Linear velocity of the vehicle (m/s)*= $\omega_W \times r_W$,

x = Wheel track (m)

b = Wheel base (m)




**(i) Reaction due to weight of Vehicle**

***Weight of the vehicle.*** Assuming that weight of the vehicle *(mg)* is equally distributed over four wheels. Therefore, the force on each wheel acting downward is m*g/4* and the reaction by the road surface on the wheel acts in upward direction.

$$R_w = \frac{mg}{4}$$

**(ii) Effect of Gyroscopic couple due to Wheel**

Gyroscopic couple due to four wheels is,

$$C_W = 4\, I_W \omega \omega_p$$

**(iii) Effect of Gyroscopic Couple due to Engine**

Gyroscopic couple due to rotating parts of the engine

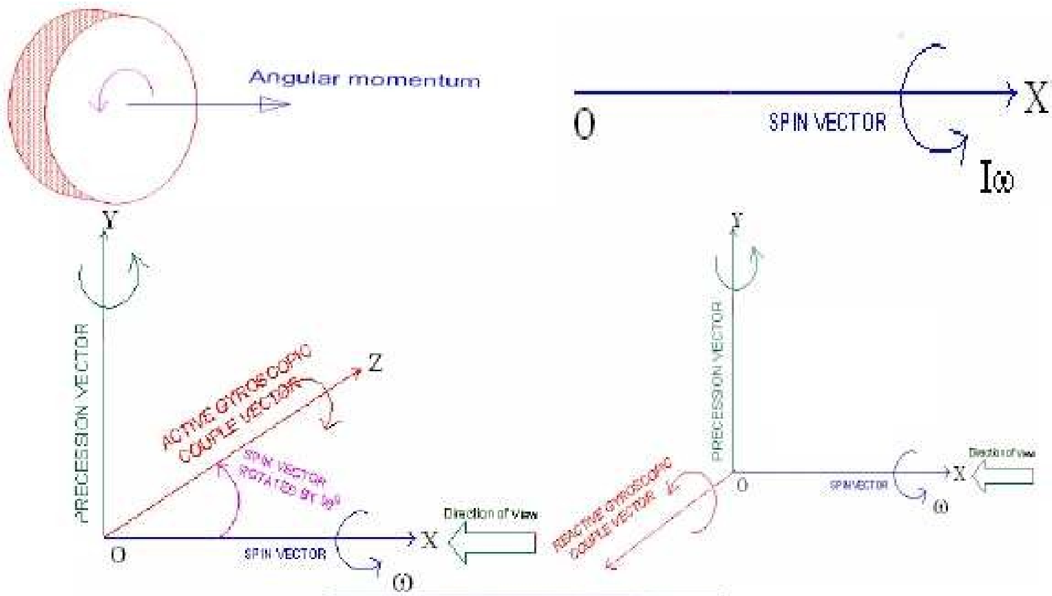
$$C_E = I_E\, \omega\, \omega_p = I_E\, G\, \omega\, \omega_p$$

Therefore, Total gyroscopic couple:

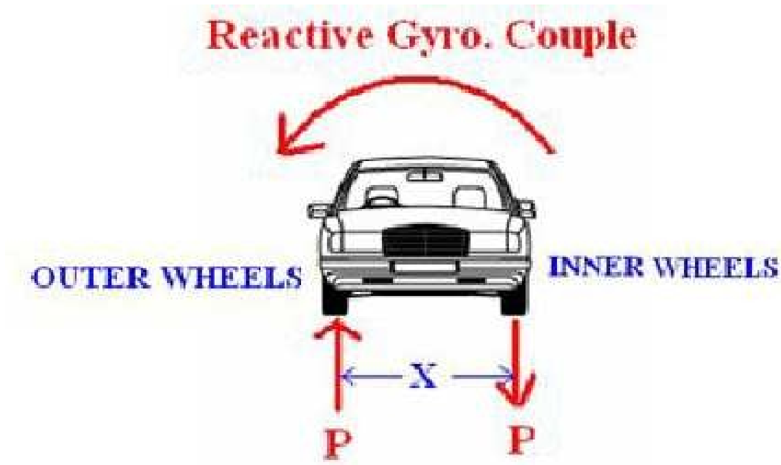
$$C_g = C_w + C_E = \omega\ \omega_p\,(4I_W \pm I_E G)$$

When the wheels and rotating parts of the engine rotate in the same direction, then positive sign is used in the above equation. Otherwise negative sign should be considered.

Assuming that the vehicle takes a left turn, the reaction gyroscopic couple on the vehicle acts between outer and inner wheels.




This gyroscopic couple tends to press the outer wheels and lift the inner wheels




Due to the reactive gyroscopic couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be P Newtons, then,

$$P \times X = C_g$$

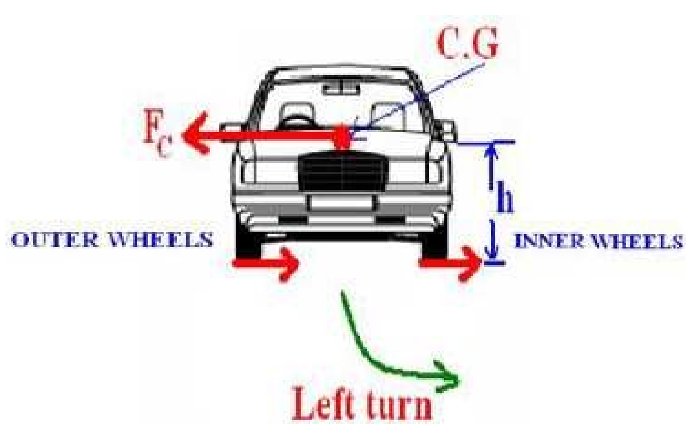
$$P = \frac{C_g}{X}$$

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_g}{2X}$$

(iii)**Effect of Centrifugal Couple**

When a vehicle moves on a curved path, a centrifugal force acts on the vehicle in outward direction through the centre of gravity of the vehicle( Fig…)



Centrifugal force,

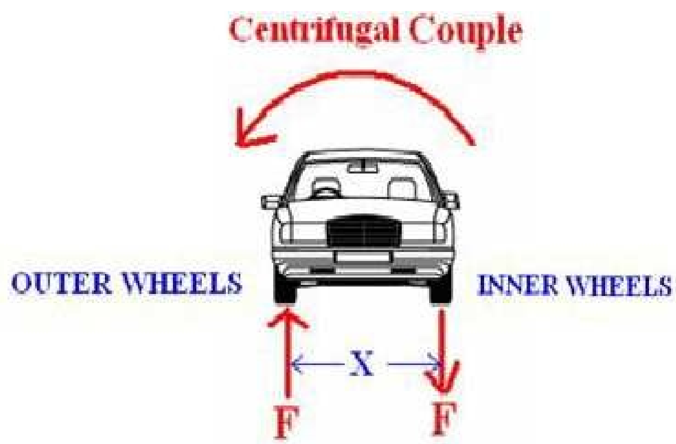
$$F_c = m\omega_p^2 R = \frac{mv^2}{R}$$

This force forms a Centrifugal couple.

$$C_c = \frac{mv^2 h}{R}$$

This centrifugal couple tends to press the outer and lift the inner

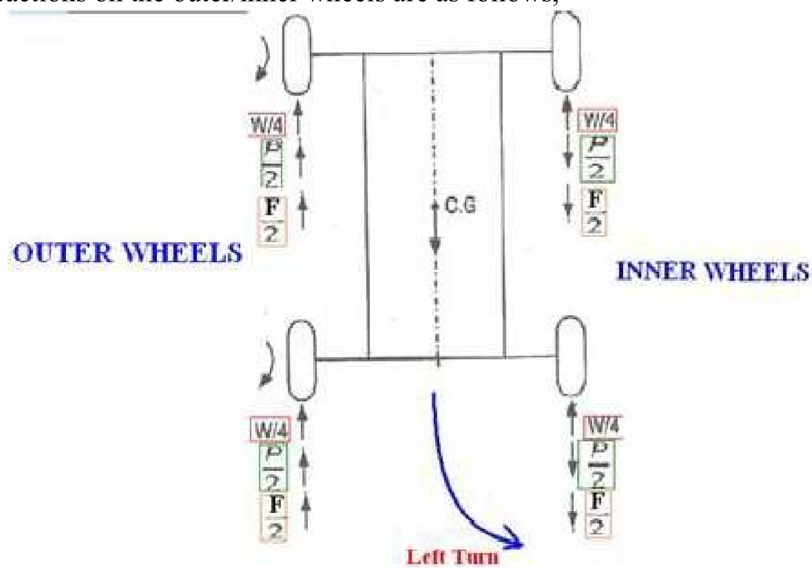
Due to the centrifugal couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be F Newtons, then,




Road reaction on each outer/Inner wheel,

$$\frac{F}{2} = \frac{C_C}{2X}$$

The reactions on the outer/inner wheels are as follows,




Total vertical reaction at each outer wheels

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

Total vertical reaction at each inner wheels

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

**MODULE-II**

# Static and Dynamic Force Analysis of Mechanisms

Mechanisms are designed to carry out certain desired work, by producing the specified motion of certain output member. It is usually required to find the force or torque to be applied on an input member. when one or more forces act on certain output member(s). The external force may be constant or varying through the whole cycle of motion. Calculation of input force or torque over the complete cycle will. be needed to determine the power requirement. then the masses and moments of inertia of the members are negliglble, static force analysis may be carried out. Otherwise, particularly at high speeds, significant forces or torques will be required to produce linear or angular accelerations of the various members. The same will have to be considered in *the* analysis. *It is .also* required *to* find the forces at the )oTnts for proper *design.* These also vary depending upon the position/configuration in the cycle.

Static analysis is carried out by the usual methods of colllnearity of forces. equlTfbrium of forces / moments. Input is determined as that force or moment to bring the system' to' equilibrium. In the case of dynamic systems, linear acceleration of each link (CG) and the angular accelerations of the members are evaluated. The corresponding forces and moments are calculated (product of acceleration and inertia).

D'Alembert's principle is a method of applying fictitious forces / torques called inertia force / torque, equal and opposite to the force or torque required to produce acceleration in each member, so as to produce a static" condition which is called dynamic equilibrium. Then the system can be treated as static, which permits application *af* techniques of static force analysis.

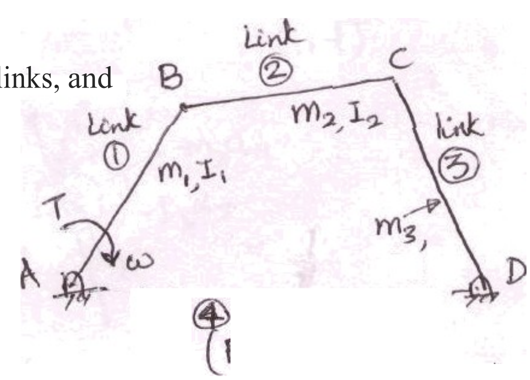
Dyriamlc force analysis is the evaluation of input forces or torques and joint forces considering motion of members. Evaluation of the inertia force /torque are explained first. Methods of static force analysis are explained.

Dynamic Force Analysis:

Consider the four-bar chain ABCD (fig.1a). Let the joint A be acted upon by a Torque T so as 'to move the link AB at ao .angular velocity of w. Let the masses of the links AB, BC and CD be m,, mz and mm, and moments of inertia be I , I,, and I›.

l. Draw the velocity {fig 1.b) and acceleration diagram (fig 1.c) of the mechanism

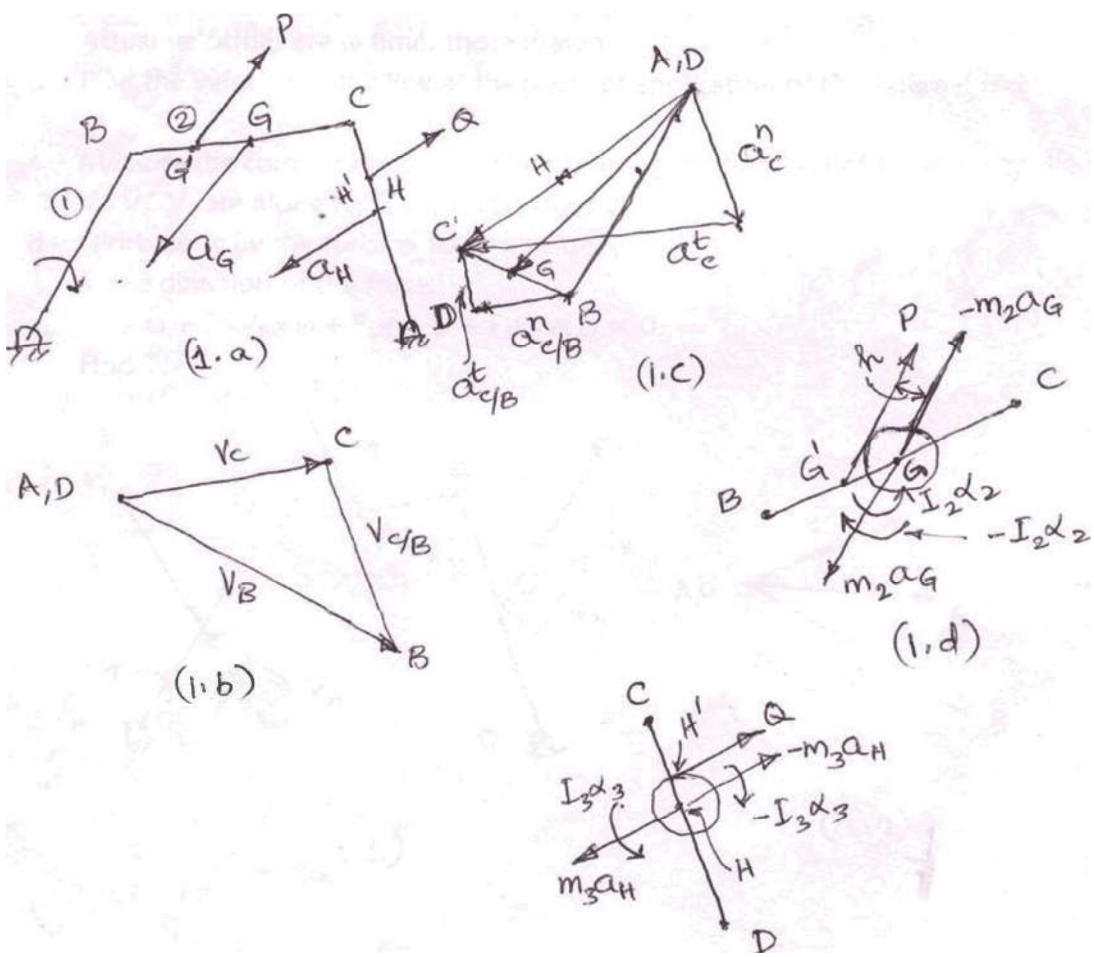
ii. Determine linear accelerations of the CGs of the links, and angular accelerations of links BC and CD.



iii. Consider link BC. Let the CG be at point G. (fig. 1.d) Force on the link due to acceleration dg is* *2 M@X0
Hence lnertia force - -fz
Angular acceleration = a› = a'› / BC;
**Torque t It x « ( )**
Inertia torque -tm (cw)

iv. Combine the inertia force and torque into a single force P, parallel to it, but acting at distance h= IQ m a2 Ite m the point G. (Fig.J.d)(Verify)
v. This force equivalently replaces the inertia force and torque.
vi. Repeat the procedure for link CD. (fig.1.e)
vii. For link AB, as there is no angular acceleration, inertia force is taken to act opposite to 1xa. (If it has finite angular acceleration. given as input, it can be handled as for other links)
viii. Thus, the mechanism will be in equilibrium under the action of the forces acting on links 2 and 3 and the input torque. It is then a static system.

The torque on the crank is calculated by any of the methods of static force analysis, some of which are explained below. *0 eT•*



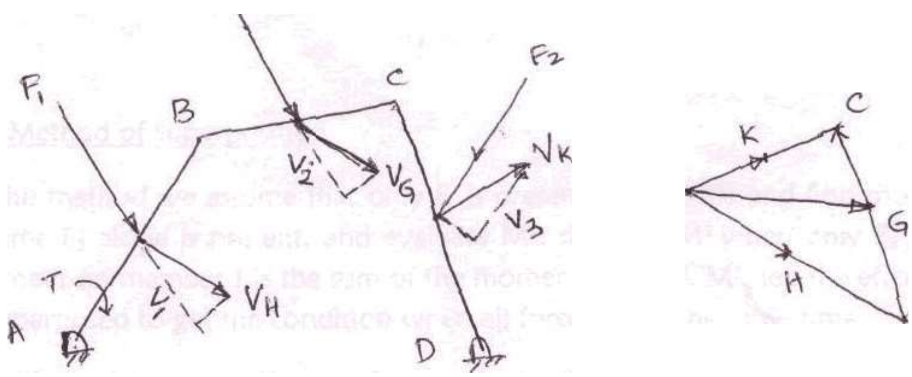
Static Force Analysts:

This can be done by obtaining the free body diagrams (f.b.d.) *for* each link, application of equilibrium of forces or moments and collinearity of forces, as appropriate. Either graphical-analytical methods or vectorial approach can be adapted. \Ve review (a) Prindple of Virtual Work (b) method of force resolution and (c) Method of superposition. {be mayalso*employ*equivalent vectorlalmethods—seeJEShigley).

ConsideraMbarchain.ForcesF,(2.F›acton links 1., 3 and 3 at the points shown: It is desired to find the torque T on link 1, (and joint forces) to keep the mechanism in equilibrium.

A.Principle of Virtual Work: In this method. total work done by forces and moment acting on the system causing infinitesimal motions, is taken as zero. It is to be noted that the reactions at.the joints get nullified and are workfess. As such the joint forces cannot be evaluated in this method. Following procedure is adapted:

a. Draw a velocity diagram of the linkage assuming unit angular velocity of the link AB on which the turning moment is applied {fig.a.i).
   Actual velocities are w times those drawn.
b. Find the velocity of the link at the point.of application of the external force.
c. Measure the component of the velocity along the direction of the force applied.
   1. Vz, V, are aloag F1. F,., F› respy. (fig.a.2)
d. Work done by the force = force x velocity in the direction of the force.
e. **T x u + F‹xVi x w + FzxVz w + F›xV›x w = 0.**
f. Find T.




2)(a.

8 : Bv Resoluion of Forces:

Start with 'link 3.

-From the fbd of link 3, let the force fr be resolved into two components, one along Link 3 and other perpendicular. (fig.b.I)

- -Take moments about D, which gives fz›' Link 2

-Fz and F32' being known, **taking moments** about B, find f3›°. (fig.b2)

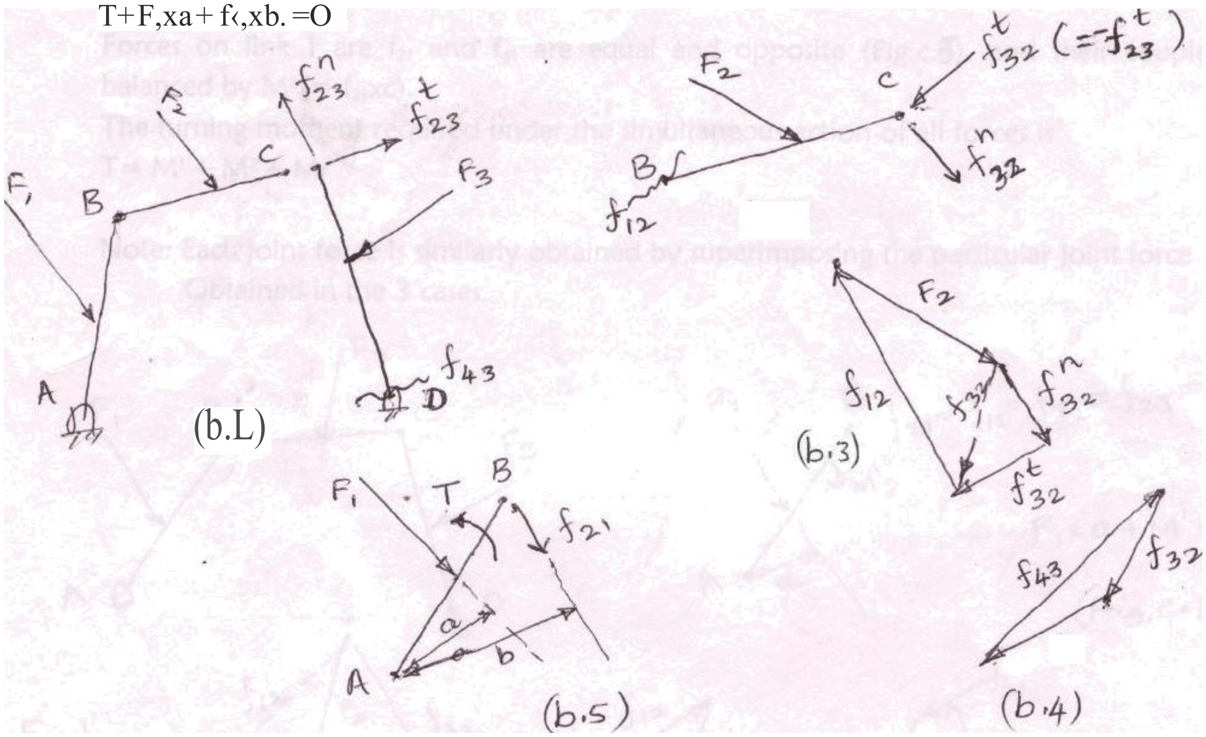
-From polygon of forces, find for (f g.b.3)

-F/ and f23 components being known, force polygon. gives fi,. (fig.b.4). Link \

*From* the polygon of forces on link i, find f<. Sko

Teking moments about A, (fig.b.5), find T from the eqn.

T+F,xa+ f‹,xb. =O




## (C) Method of Superposition

In this method we assume that only F is present (Fm, F› -0) and find moment M. Then assume Ft alone is present, **and evaluate M', similarly** M* when only Fm is present. The moment on member \Is the su.m of the moments M[1],M°.M*. ie., the effect of each force **is superposed to** get the condition when all forces act at the same time.

fa) Effect of .F1 alone (fig.c.i): Start wlth the fbd for link 1 - links 2 and 3 are 2-force members, and joint forces are along the members. However, at joint C, force f‹, and f›, act along the respective members 2 and 3, but have to be: equal *and* opposite. "It is possible only fzz=f›z=0. Hence, f2 f= fn) and f,‹(=ft3t I! all be zero.

Fi and f‹ are equal and opposite. The moment F,xa is balanced by M’. (M’ + F xa =0)

(b) F alone acting: From the fbd of link2- Forces Fz, ! 32 (along link 3. beirtg 2-force member) are collinear, which determines the direction of t2 fi c Now complete the force polygon to determine the magnitudes of f,2 and /›2 as well. (fig.c.3). Also, f›3=f••

On link 1. /4› and fm are equal and opposite, and balanced by M* given by M’ + f,,xb =0.

@ Force F3 on Link 3 alone (Fie. : Consider fbd of link 3. F3. f2› and f» are collinear. from which directions *ol* ftt and f,, are known. Their magnitudes are known from force P !Y8 • /32 ( f,,) are the forces acting on link 2.

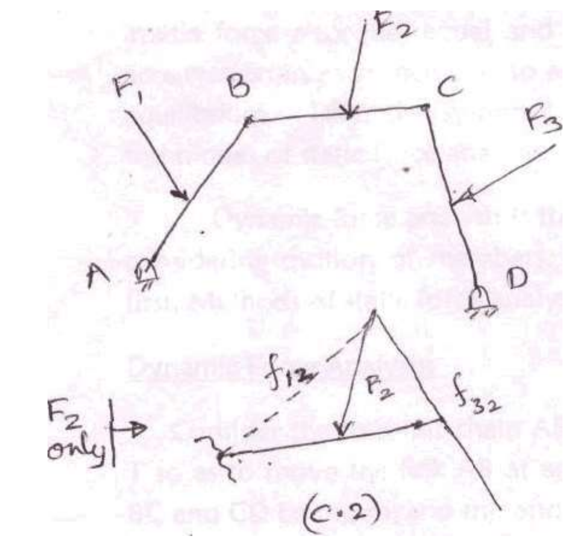
Forces on link 1 are t2 and f4 . are equal and opposite (Fig.c.@, and their couple is balanced by M*(= fmmxc).

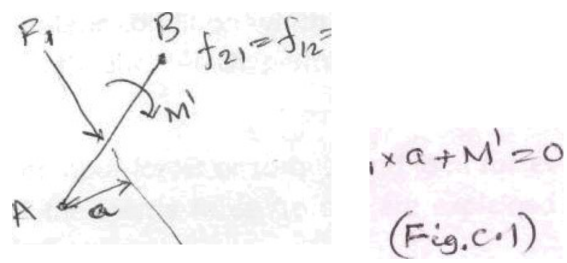
The turning moment required under the simultaneous action of all forces is

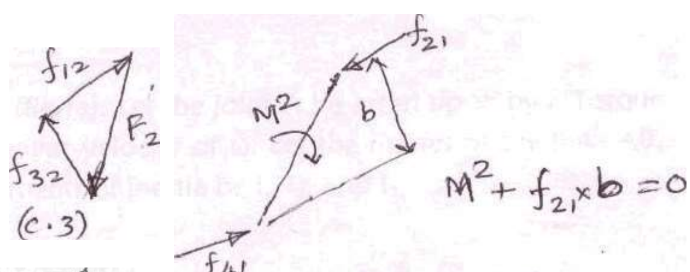
T= *M* + *M›* + M’

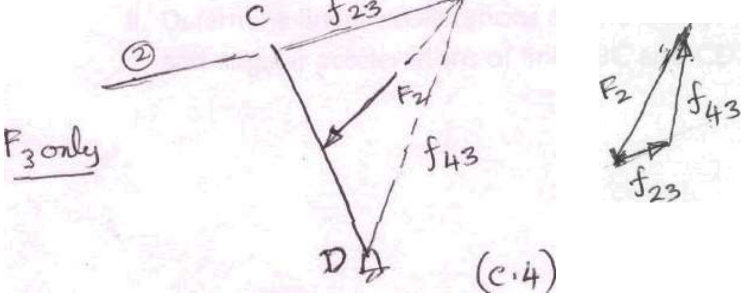
Note: Each joint force is similarly obtained by superimposing the particular joint force

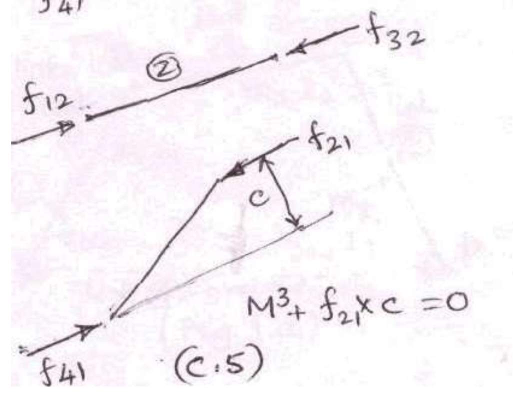
Obtained in the 3 cases.



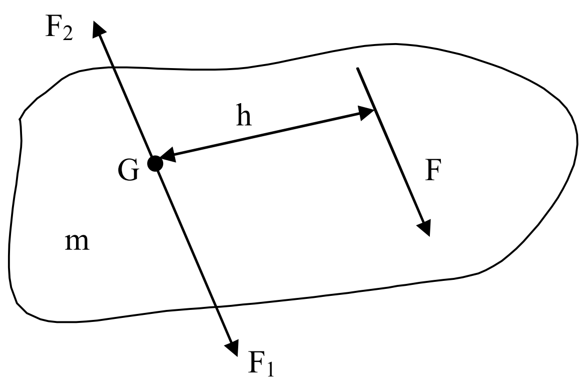
**DYNAMIC FORCE ANALYSIS:**

It is defined as the study of the force at the pin and guiding surfaces and the forces causing stresses in machine parts, such forces being the result of forces due to the motion of each part in the machine. The forces include both external and inertia forces. Inertia forces in high speed machines become very large and cannot be neglected, Ex: Inertia force of the piston of an automobile travelling at high speed might be thousand times the weight of the piston. The dynamic forces are associated with accelerating masses.

If each link, with its inertia force and force applied to the link can be considered to be in equilibrium, the entire system can also be considered to be in equilibrium.

## Determination of force & couple of a link

(Resultant effect of a system of forces acting on a rigid body)




G = c .g point
$F_1$& $F_2$: equal and opposite forces acting through G (Parallel to F)

F: Resultant of all the forces acting on the rigid body.
h: perpendicular distance between F & G.
m = mass of the rigid body

**Note:** $F_1$=$F_2$ & opposite in direction; they can be cancelled with out affecting the equilibrium of the link. Thus, a single force „F" whose line of action is not through G, is capable of producing both linear & angular acceleration of CG of link.

F and $F_2$ form a couple.

T= F x h = I α = mk$^2$ α (Causes angular acceleration) ..............(1)

Also, $F_1$ produces linear acceleration, f.

$F_1$= mf

Using 1 & 2, the values of „f" and „α" can be found out if $F_1$, m, k & h are known.

**D’Alembert’s principle:**

Final design takes into consideration the combined effect of both static and dynamic force systems. D"Alembert"s principle provides a method of converting dynamics problem into a static problem.

***Statement:***

The vector sum of all external forces and inertia forces acting upon a rigid body is zero. The vector sum of all external moments and the inertia torque, acting upon the rigid body is also separately zero.

In short, sum of forces in any direction and sum of their moments about any point must be zero.

***Inertia force and couple:***

Inertia: Tendency to resist change either from state of rest or of uniform motion

Let „R" be the resultant of all the external forces acting on the body, then this „R" will be equal to the product of mass of the body and the linear acceleration of c.g of body. The force opposing this „R" is the inertia force (equal in magnitude and opposite in direction).

*(Inertia force is an Imaginary force equal and opposite force causing acceleration)*

If the body opposes angular acceleration (α) in addition to inertia force R, at its cg, there exists an inertia couple Ig x α, Where Ig= M I about cg. The sense of this couple opposes α. i.e., inertia force and inertia couple are equal in magnitude to accelerating force and couple respectively but, they act in opposite direction.

Inertia force ($F_i$) = M x f,
(mass of the rigid body x linear acceleration of cg of body)

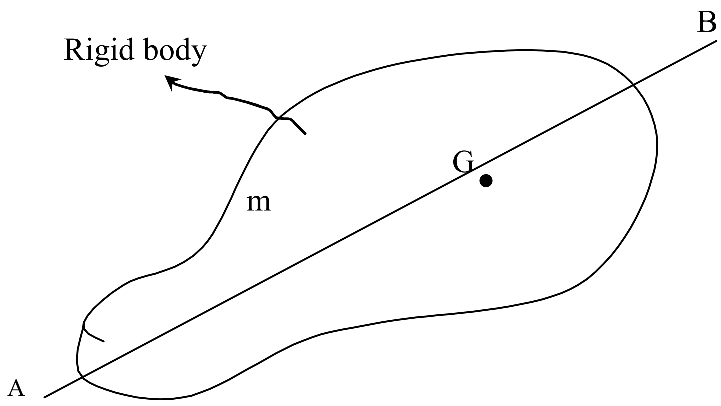
Inertia couple ($C_i$)=I x α, MMI of the rigid body about an axis perpendicular to the plane of motion | Angular acceleration

***Dynamic Equivalence:***

The line of action of the accelerating force can also be determined by replacing the given link by a dynamically equivalent system. Two systems are said to be dynamically equivalent to one another, if by application of equal forces, equal linear and angular accelerations are produced in the two systems.

i.e., the following conditions must be satisfied;

i) The masses of the two systems must be same.
ii) The cg"s of the two systems must coinside.
iii) The moments of inertia of the two systems about same point must be equal, Ex: about an axis through cg.

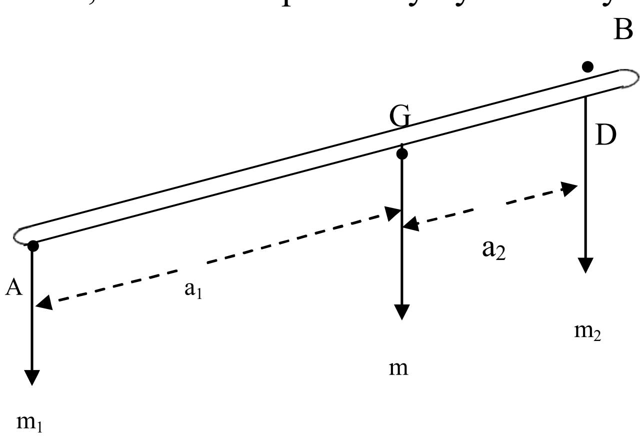



G = c.g.

m = mass of the rigid body

$k_g$ = radius of gyration about an axis through G and perpendicular to the plane

Now, it is to be replaced by dynamically equivalent system.




$m_1, m_2$ – masses of dynamically equivalent system at $a_1$ & $a_2$ from G (respectively)

As per the conditions of dynamic equivalence,

$$m = m_1 + m_2 \quad .. (a)$$
$$m_1 a_1 = m_2 a_2 \quad .. (b)$$
$$mk_g^2 = m_1 a_1^2 + m_2 a_2^2 \quad .. (c)$$

Substituting (b) in (c),

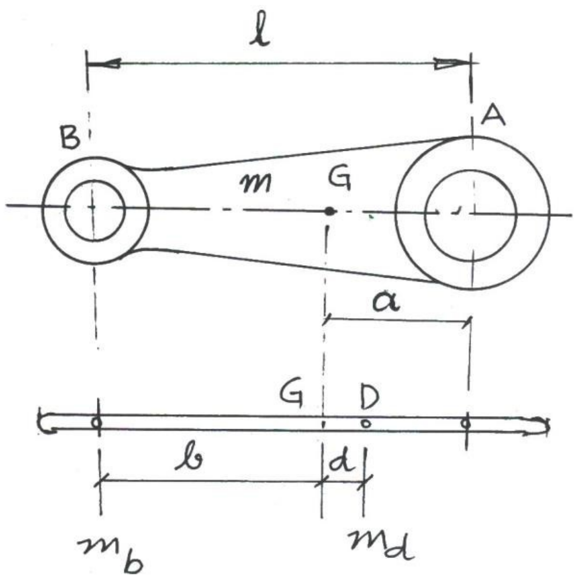
$$mk_g^2 = (m_2 a_2) a_1 + (m_1 a_1) a_2$$

$$= a_1 a_2 (m_2 + m_1) = a_1 a_2 (m)$$

i.e., $k_g^2 = a_1 a_2$ $\quad [I_g = mk_g^2 \text{ or } k_g^2 = \frac{I_g}{m}]$

or $\frac{I_g}{m} = a_1 a_2$

## *Inertia of the connecting rod:*




Connecting rod to be replaced by a massless link with two point masses $m_b$ & $m_d$.

m = Total mass of the CR $m_b$ & $m_d$ point masses at B& D.

$$m_b + m_d = m \qquad -- \quad (i)$$

$$m_b \times b = m_d \times d \qquad -- (ii)$$

Substituting (ii) in (i);

$$m_b + \left( m_b \times \frac{b}{d} \right) = m$$

$$m_b \left( 1 + \frac{b}{d} \right) = m \qquad \text{or } m_b \left( \frac{b+d}{d} \right) = m$$

$$\text{or } m_b = m \left( \frac{d}{b+d} \right) \qquad -- (1)$$

Similarly;

$$m_d = m \left( \frac{b}{b+d} \right) \qquad -- (2)$$

Also; $I = m_b b^2 + m_d d^2$

$$= m \left( \frac{d}{b+d} \right) b^2 + m \left( \frac{b}{b+d} \right) d^2 \qquad [\textit{from } (1) \ \& \ (2)]$$

$$I = mbd \left( \frac{b+d}{b+d} \right) = mbd$$

Then, $mk_g^2 = mbd$, (since $I = mk_g^2$)

$$k_g^2 = bd$$

The result will be more useful if the 2 masses are located at the centers of bearings A & B.

Let $m_a$ = mass at A and dist. AG = a

Then,

$$m_a + m_b = m$$

$$m_a = m\left(\frac{b}{a+b}\right) = m\frac{b}{l}\;; \qquad Since\ (a+b=l)$$

Similarly, $$m_b = m\left(\frac{a}{a+b}\right) = m\frac{a}{l}\;; \qquad (Since,\ a+b=l)$$

$$I^1 = m_a a^2 + m_b b^2 = \ldots \ldots = mbd$$ (Proceeding on similar lines it can be proved)

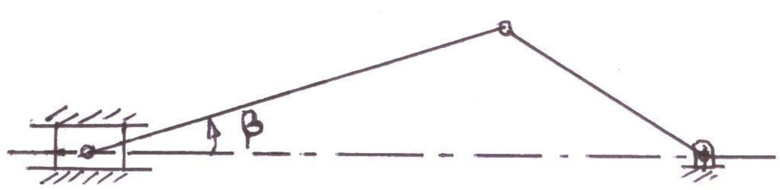
Assuming; $a > d,\ I^1 > I$

i.e., by considering the 2 masses A & B instead of D and B, the inertia couple (torque) is increased from the actual value. i.e., there exists an error, which is corrected by applying a correction couple (opposite to the direction of applied inertia torque).

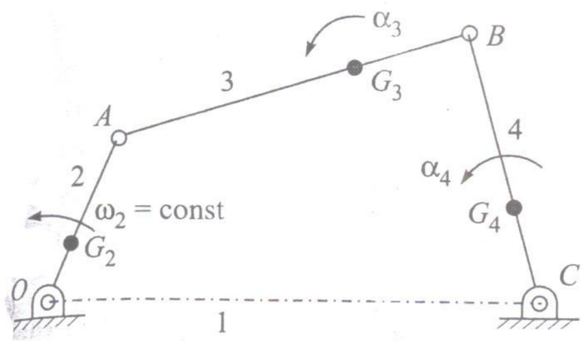
The correction couple,

$$\Delta T = \alpha_c\,(mab - mbd)$$
$$= mb\,\alpha_c\,(a-d)$$
$$= mb\,\alpha_c\left[\,(a+b)-(b+d)\right]$$
$$= mb\,\alpha_c\,(l-L) \qquad because\ (b+d=L)$$

As the direction of applied inertia torque is always opposite to the direction of angular acceleration, the direction of the correction couple will be same as that of angular acceleration i.e., in the direction of decreasing angle β.



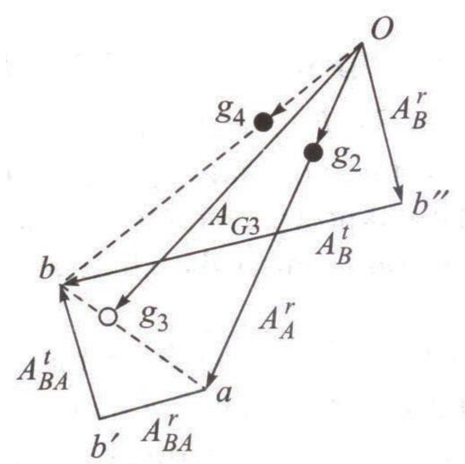
**Dynamic force Analysis of a 4 – link mechanism.**




OABC is a 4–bar mechanism. Link 2 rotates with constant $\omega_2$. $G_2$, $G_3$ & $G_4$ are the cgs and $M_1$, $M_2$ & $M_3$ the masses of links 1, 2 & 3 respectively.

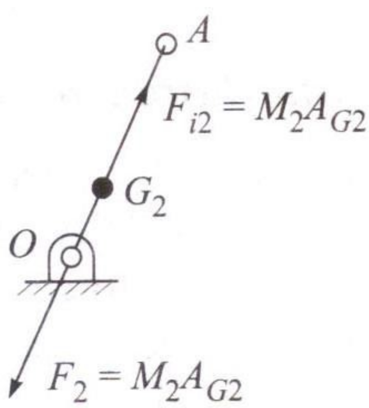
What is the torque required, which, the shaft at o must exert on link 2 to give the desired motion?

1. Draw the velocity & acceleration polygons for determing the linear acceleration of $G_2$, $G_3$ & $G_4$.
2. Magnitude and sense of $\alpha_3$ & $\alpha_4$ (angular acceleration) are determined using the results of step 1.




**To determine inertia forces and couples**

**Link 2**

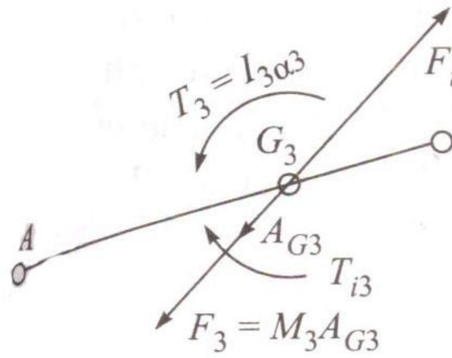



$F_2$ = accelerating force (towards O)

$F_{i2}$ = inertia force (away from O)

Since $\omega_2$ is constant, $\alpha_2 = 0$ and no inertia torque involved.

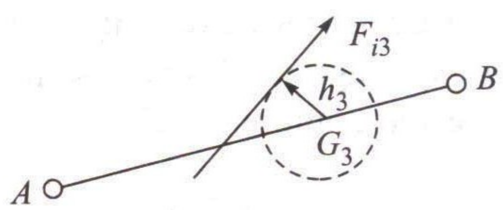
***Link 3***




Linear acceleration of $G_3$ (i.e., $AG_3$) is in the direction of $Og_3$ of acceleration polygon.

$F_3$ = accelerating force

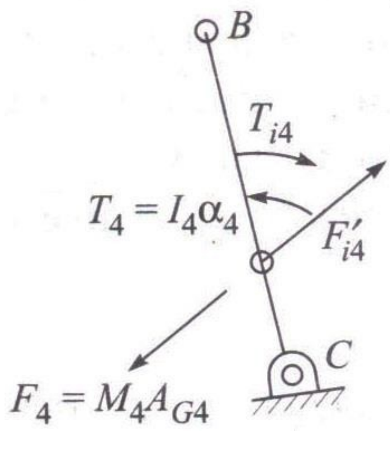
Inertia force $F_{i3}^{'}$ acts in opposite direction. Due to $\alpha_3$, there must be a resultant torque $T_3 = I_3\ \alpha_3$ acting in the sense of $\alpha_3$ ($I_3$ is MMI of the link about an axis through $G_3$, perpendicular to the plane of paper). The inertia torque $T_{i3}$ is equal and opposite to $T_3$.

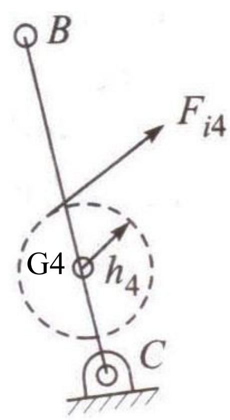



$F_{i3}$ can replace the inertia force $F_{i3}^{'}$ and inertia torque $T_{i3}$. $F_{i3}$ is tangent to circle of radius $h_3$ from $G_3$, on the top side of it so as to oppose the angular acceleration $\alpha_3$.

$$h3 = \frac{I_3\alpha_3}{M_3 AG_3}$$

**Link 4**




$$h4 = \frac{I_4\alpha_4}{M_4 AG_4}$$

**Problem 1 :**

It is required to carryout dynamic force analysis of the four bar mechanism shown in the figure.

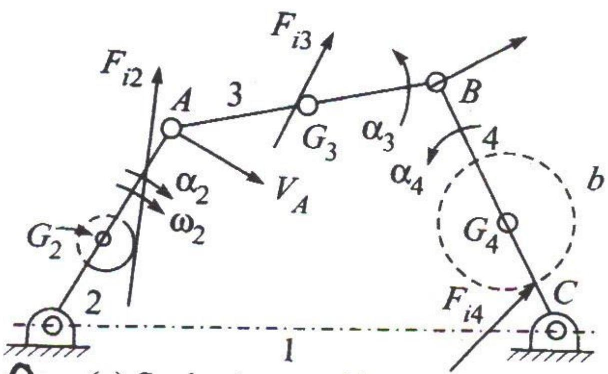
$\omega_2$ = 20rad /s (cw), $\alpha_2$ = 160 rad/s$^2$ (cw)

OA= 250mm, $OG_2$= 110mm, AB=300mm, $AG_3$=150mm, BC=300mm, $CG_4$=140mm, OC=550mm, $\angle AOC = 60°$

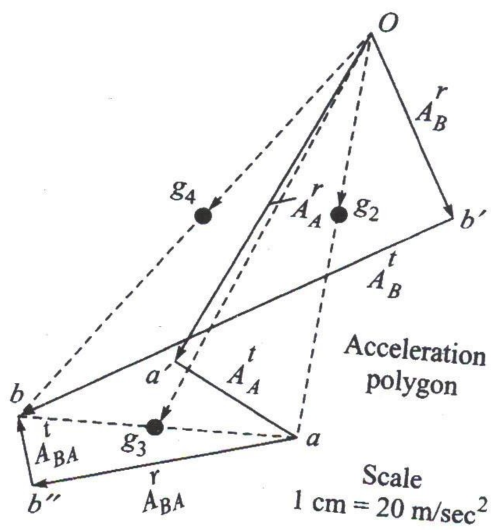
The masses & MMI of the various members are

| Link | Mass, m | MMI ($I_G$, $Kgm^2$) |
|---|---|---|
| 2 | 20.7kg | 0.01872 |
| 3 | 9.66kg | 0.01105 |
| 4 | 23.47kg | 0.0277 |

Determine i) the inertia forces of the moving members
ii) Torque which must be applied to (2)



(a) Scale: 1 cm = 10 cms




## A) Inertia forces:

***(i) (from velocity & acceleration analysis)***

$V_A = 250\times 20;\ 5m/s,\quad V_B = 4\ m/s,\quad V_{BA} = 4.75\ m/s$

$a_A^r = 250\times 20^2;100m/s^2,\quad a_A^t = 250\times 160;\ 40m/s^2$

Therefore;

$$A_B^r = \frac{V_B^2}{CB} = \frac{(4)^2}{0.3} = 53.33\ m/s^2$$

$$A_{BA}^r = \frac{V_{BA}^2}{B_A} = \frac{(4.75)^2}{0.3} = 75.21\ m/s^2$$

$$Og_2 = A_{G2} = 48\ m/s^2;\ Og_3 = AG_3 = 120m/s^2$$

$$Og_4 = A_{G4} = 65.4\ m/s^2$$

$$\alpha_3 = \frac{A^t_{BA}}{AB} = \frac{19}{0.3} = 63.3\ rad/s^2$$

$$\alpha_4 = \frac{A^t_B}{CB} = \frac{129}{0.3} = 430\ rad/s^2$$

***Inertia forces (accelerating forces)***

$$F_{G2} = m_2 \; A_{G2} = \frac{20.7}{9.81} \times 48 = 993.6 \, N \; (\textit{ in thedirection of } Og_2)$$

$$F_{G3} = m_3 \; A_{G3} = 9.66 \times 120 = 1159.2 \, N \; (\textit{ in the direction of } Og_3)$$

$$= F_{G4} = m_4 A_{G4} = 23.47 \times 65.4 = 1534.94 \, N \; (\textit{in thedirection of } Og_4)$$

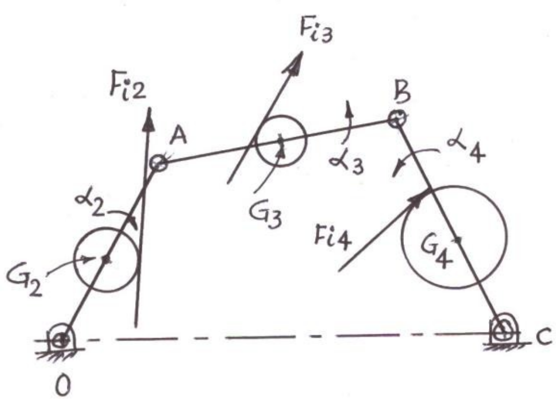
$$h_2 = \frac{I_{G2}(\alpha_2)}{F_2} = \frac{(0.01872 \times 160)}{993.6} = 3.01 \times 10^{-3} \, m$$

$$h_3 = \frac{I_{G3}(\alpha_3)}{F_3} = \frac{(0.01105 \times 63.3)}{1159.2} = 6.03 \times 10^{-4} \, m$$

$$h_4 = \frac{I_{G4}(\alpha_4)}{F_4} = \frac{(0.0277 \times 430)}{1534.94} = 7.76 \times 10^{-3} \, m$$

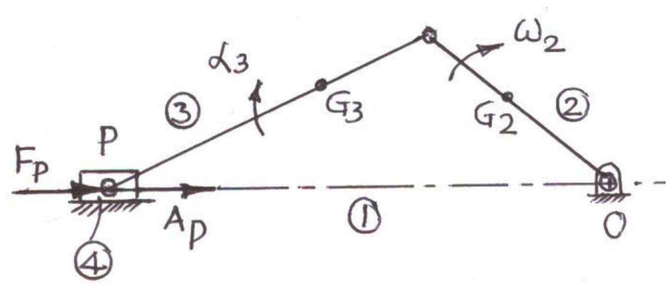
The inertia force $F_{i2}$, $F_{i3}$ & $F_{i4}$ have magnitudes equal and direction opposite to the respective accelerating forces and will be tangents to the circles of radius $h_2$, $h_3$ & $h_4$ from $G_2$, $G_3$ & $G_4$ so as to oppose $\alpha_2$, $\alpha_3$ & $\alpha_4$.

$F_{i2} = 993.6 \, N$ , $F_{i3} = 1159.2 \, N$ $F_{i4} = 1534.94 N$




Further, each of the links is analysed for static equilibrium under the action of all external force on that link plus the inertia force.

## Dynamic force analysis of a slider crank mechanism.




$F_p$ = load on the piston

| Link | mass | MMI |
|---|---|---|
| 2 | $m_2$ | $I_2$ |
| 3 | $m_3$ | $I_3$ |
| 4 | $m_4$ | - |

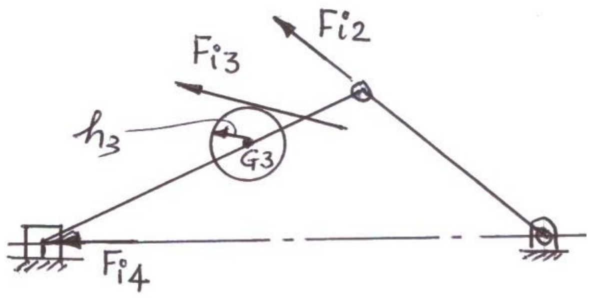
$\omega_2$ assumed to be constant

***Steps involved:***

1. Draw velocity & acceleration diagrams
2. Consider links 3 & 4 together and single FBD written (elimination $F_{34}$ & $F_{43}$)
3. Since, weights of links are smaller compared to inertia forces, they are neglected unless specified.
4. Accelerating forces $F_2$ , $F_3$ & $F_4$ act in the directions of respective acceleration vectors $Og_2$, $Og_3$ & $Og_p$

Magnitudes: $F_2 = m_2 AG_2$ $F_3 = m_3 AG_3$ $F_4 = m_4 A_p$

$F_{i2} = F_2$ , $F_{i3} = F_3$ , $F_{i4} = F_4$ (Opposite in direction)



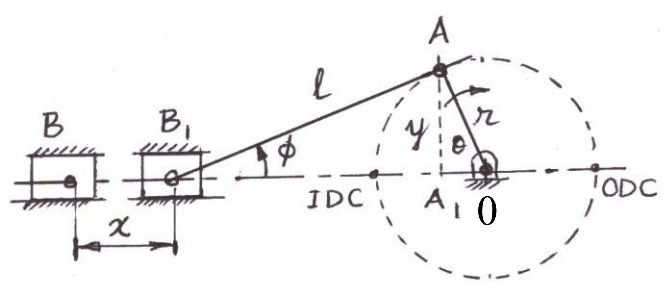

$$h_3 = \frac{I_3 \alpha_3}{M_3 \alpha_{g3}}$$

$F_{i3}$ is tangent to the circle with $h_3$ radius on the RHS to oppose $\alpha_3$

Solve for $T_2$ by solving the configuration for both static & inertia forces.

# Dynamic Analysis of slider crank mechanism (Analytical approach)

***Displacement of piston***




$x$ = displacement from IDC

$x = BB_1 = BO - B_1O$

$= BO - (B_1A_1 + A_1O)$

$= (l + r) - (l\cos\phi + r\cos\theta)$ $\left(\text{since}, \frac{l}{r} = n\right)$

$= (nr + r) - (rn\cos\phi + r\cos\theta)$

$= r\left[(n+1) - (n\cos\phi + \cos\theta)\right]$ $\cos\phi = \sqrt{1 - \sin^2\phi}$

$$= r\left[(n+1) - \left(\sqrt{n^2 - \sin^2\theta} + \cos\theta\right)\right]$$

$$= r\left[(1-\cos\theta) + \left(n - \sqrt{n^2 - \sin^2\theta}\right)\right]$$

$$= \sqrt{1 - \frac{y^2}{l^2}}$$

$$= \sqrt{1 - \frac{(r\sin\theta)^2}{l^2}}$$

$$= \sqrt{1 - \frac{\sin^2\theta}{n^2}}$$

$$= \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

(*similary* $l >> r, \frac{l}{r} = n >> 1$ & max *value of* $\sin\theta = 1$)

$\therefore \sqrt{n^2 - \sin^2\theta} \rightarrow \sqrt{n^2}$ *or* $n$),

$$\boxed{x = r\,(1-\cos\theta)}$$

This represents SHM and therefore Piston executes SHM.

### *Velocity of Piston:*

$$v = \frac{dx}{dt} = \frac{dx}{d\theta}\,\frac{d\theta}{dt}$$

$$\frac{d}{d\theta}\left[r\,(1-\cos\theta) + n - (n^2 - \sin 2\theta)^{-\frac{1}{2}}\right]\frac{d\theta}{dt}$$

$$= r\left[0 + \sin\theta + 0 - \frac{1}{2}(n^2 - \sin 2\theta)^{-1/2}(-2\sin\theta\cos\theta)\right]\omega$$

$$= r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right]$$

Since, $n^2 >> \sin^2\theta$,

$$\therefore\ v = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]$$

Since n is quite large, $\frac{\sin 2\theta}{2n}$ can be neglected.

$$\boxed{\therefore\ v = r\omega\sin\theta}$$

## *Acceleration of piston:*

$$a = \frac{dv}{dt} = \frac{dv}{d\theta}\,\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left[ r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right)\omega \right]$$

$$= r\omega\left[\cos\theta + \frac{2\cos 2\theta}{2n}\right]$$

$$= r\omega\left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

If n is very large;

$$\boxed{a = r\omega^2\cos\theta} \quad \textbf{(as in SHM)}$$

When $\theta = 0$, at IDC,

$$a = r\omega^2\left(1 + \frac{1}{n}\right)$$

When $\theta = 180$, at 0DC,

$$a = r\omega^2\left(-1 + \frac{1}{n}\right)$$

At $\theta = 180$, when the direction is reversed,

$$a = r\omega^2\left(1 - \frac{1}{n}\right)$$

**Angular velocity & angular acceleration of CR ($\alpha_c$)**

$$y = l\sin\phi = r\sin\theta$$

$$\sin\phi = \frac{\sin\theta}{n}$$

Differentiating w.r.t time,

$$\cos\phi\,\frac{d\phi}{dt} = \frac{1}{n}\cos\theta\,\frac{d\theta}{dt} \qquad \frac{d\phi}{dt} = \omega_c$$

$$\omega_c = \omega\frac{\cos\theta}{n\frac{1}{n}\sqrt{n^2 - \sin^2\theta}} \qquad \frac{d\theta}{dt} = \omega$$

$$\cos\phi = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$$\omega_c = \omega \frac{\cos\theta}{\sqrt{n^2 - \sin^2}}$$

$$\alpha_c = \frac{d\omega_c}{dt} = \frac{d\omega_c}{d\theta}\frac{d\theta}{dt}$$

$$= \omega \frac{d}{d\theta}\left[\cos\theta\,(n^2 - \sin^2\theta)^{-\frac{1}{2}}\right]\omega$$

$$= \omega^2 \left[\cos\theta - \frac{1}{2}(n^2 - \sin^2\theta)^{-\frac{3}{2}}(-2\sin\theta\cos\theta) + (n^2 - \sin^2\theta)^{-\frac{1}{2}}(-\sin\theta)\right]$$

$$= \omega^2 \sin^2\theta \left[\frac{\cos^2\theta - (n^2 - \sin^2\theta)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}}\right]$$

$$= -\,\omega^2 \sin\theta \left[\frac{(n^2 - 1)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}}\right]$$

Negative sign indicates that, $\phi$ reduces (in the case, the angular acceleration of CR is CW)

# Engine force Analysis:

Forces acting on the engine are weight of reciprocating masses & CR, gas forces, Friction & inertia forces (due to acceleration & retardation of engine elements)

### *i) Piston effort (effective driving force)*

- Net or effective force applied on the piston.

## In reciprocating engine:

The reciprocating parts (masses) accelerate during the first half of the stroke and the inertia forces tend to resist the same. Thus, the net force on the piston is reduced. During the later half of the stroke, the reciprocating masses decelerate and the inertia forces oppose this deceleration or acts in the direction of applied gas pressure and thus effective force on piston is increased.

In vertical engine, the weights of the reciprocating masses assist the piston during out stroke (down) this in creasing the piston effort by an amount equal to the weight of the piston. During the in stroke (up) piston effect is decreased by the same amount.

Force on the piston due to gas pressure; $F_P = P_1 A_1 - P_2$ $P_1$ = Pressure

on the cover end, $P_2$ = Pressure on the rod

$A_1$ = area of cover end, $A_2$ = area of rod end, m = mass of the reciprocating parts.

Inertia force ($F_i$) = m a

$$= m.r\omega^2\left(Cos\theta + \frac{Cos2\theta}{n}\right)$$ (Opposite to acceleration of piston)

Force on the piston F = $F_P$ - $F_i$

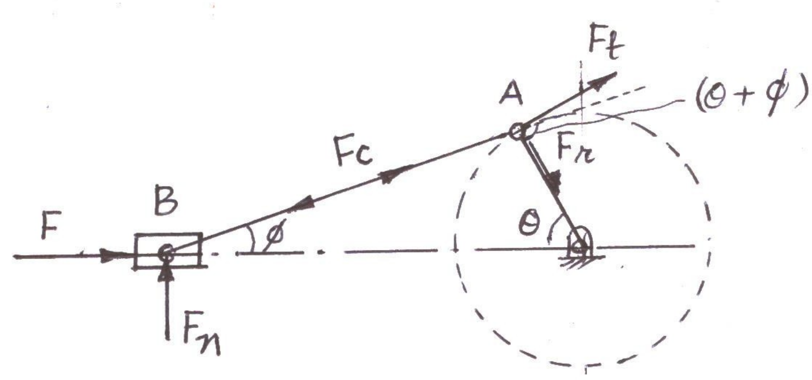
(if $F_f$ frictional resistance is also considered) F = $F_P$ – $F_i$

$-F_i$

In case of vertical engine, weight of the piston or reciprocating parts also acts as force.

$\therefore$ F = $F_P$ + mg – $F_i$ – $F_i$

### *ii) Force (Thrust on the CR)*




$F_c$ = force on the CR

Equating the horizontal components;

$$F_c\ Cos\phi = F \ \ or\ \ F_c\ \frac{F}{Cos^2\phi}$$

### *iii) Thrust on the sides of the cylinder*

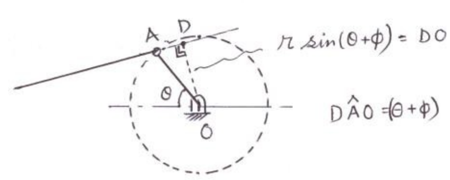
It is the normal reaction on the cylinder walls

$$F_n = F_c \sin\phi = F\tan\phi$$

### *iv) Crank effort (T)*

It is the net force applied at the crank pin perpendicular to the crank which gives the required TM on the crank shaft.

$$F_t \times r = F_c r\sin(\theta+\phi)$$




$$F_t = F_c\sin(\theta+\phi)$$

$$= \frac{F}{\cos\phi}\sin(\theta+\phi)$$

## *v) Thrust on bearings ($F_r$)*

The component of $F_C$ along the crank (radial) produces thrust on bearings

$$F_r = F_c \quad Cos(\theta + \phi) = \frac{F}{Cos\ \phi}\ Cos(\theta + \phi)$$

## *vi) Turning moment of Crank shaft*

$$T = F_t \times r$$

$$= \frac{F}{\cos\phi}\sin(\theta+\phi) \times r \ = \frac{F_r}{\cos\phi}\ (\sin\ \theta + \cos\phi + \cos\theta \sin\ \phi)$$

$$= F \times r \left( \sin\theta + \cos\theta\ \frac{\sin\ \phi}{\cos\phi} \right)$$

Proved earlier

$$= F \times r \left( \sin\ \theta + \cos\theta\ \frac{\sin\theta}{n}\ \frac{1}{\frac{1}{n}\sqrt{n^2 - \sin^2\theta}} \right)$$

$$\cos\phi = \frac{1}{n}\sqrt{n^2 - \sin^2 6}$$

$$\sin\ \phi = \frac{\sin\ \theta}{n}$$

$$= F \times r \left( \sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$

Also,

$$r\ \sin(\theta + \phi) = OD\ \cos\phi$$

$$T = F_t \times r$$

$$= \frac{F}{\cos\phi}.\ r\ \sin\ (\theta + \phi)$$

$$= \frac{F}{\cos\phi}.\ OD \quad \cos\phi$$

$$T = F \times OD.$$

# UNIT –III

## Clutches, Brakes& Dynamometers

### Friction Clutches

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently. Its application is also found in cases in which power is to be delivered to machines partially or fully loaded. The force of friction is used to start the driven shaft from rest and gradually brings it up to the proper speed without excessive slipping of the friction surfaces. In automobiles, friction clutch is used to connect the engine to the driven shaft. In operating such a clutch, care should be taken so that the friction surfaces engage easily and gradually brings the driven shaft up to proper speed. The proper alignment of the bearing must be maintained and it should be located as close to the clutch as possible. It may be noted that

**1.** The contact surfaces should develop a frictional force that may pick up and hold the load with reasonably low pressure between the contact surfaces.

**2.** The heat of friction should be rapidly dissipated and tendency to grab should be at a minimum.

**3.** The surfaces should be backed by a material stiff enough to ensure a reasonably uniform distribution of pressure.

The friction clutches of the following types are important from the subject point of view :

**1.** Disc or plate clutches (single disc or multiple disc clutch),

**2.** Cone clutches, and

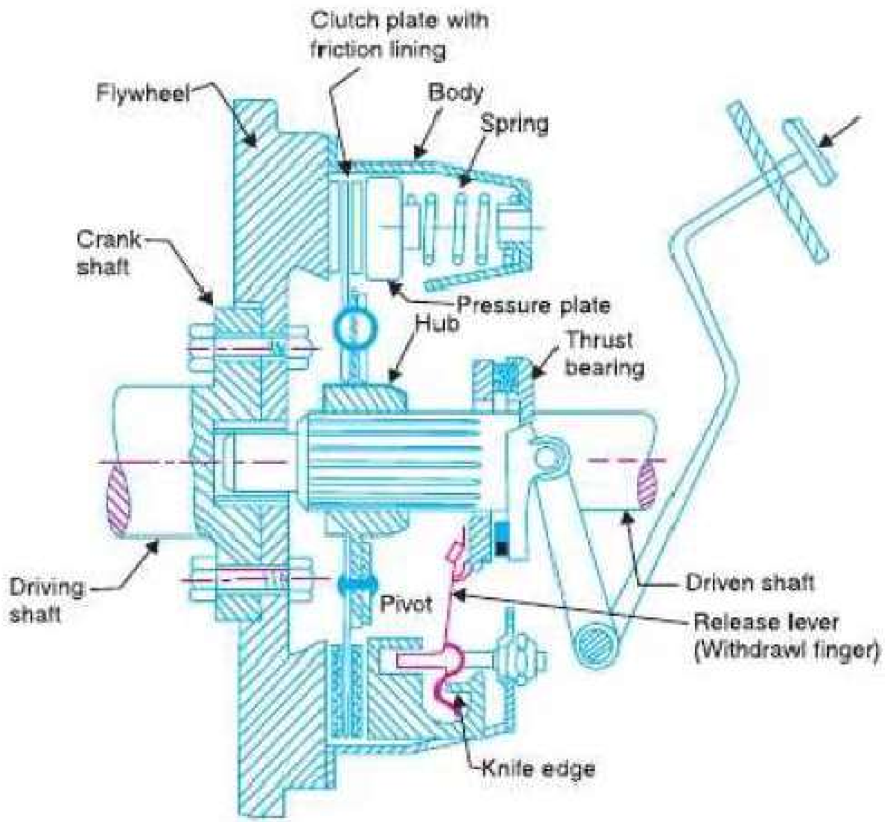
**3.** Centrifugal clutches.

We shall now discuss, these clutches, in detail, in the following pages. It may be noted that the disc and cone clutches are based on the same theory as the pivot and collar bearings.

### Single Disc or Plate Clutch

A single disc or plate clutch, as shown in Fig. 10.21, consists of a clutch plate whose both sides are faced with a friction material (usually of Ferrodo). It is mounted on the hub which is free to move axially along the splines of the driven shaft. The pressure plate is mounted inside the clutch body which is bolted to the flywheel. Both the pressure plate and the flywheel rotate with the engine crankshaft or the driving shaft. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs which are arranged radially inside the body. The three levers (also known as release levers or fingers) are carried on pivots suspended from the case of the body. These are arranged in such a manner so that the pressure plate moves away from the flywheel by the inward movement of a thrust bearing. The bearing is mounted upon a forked shaft and moves forward when the clutch pedal is pressed.

When the clutch pedal is pressed down, its linkage forces the thrust release bearing to move in towards the flywheel and pressing the longer ends of the levers inward. The levers are forced to turn on their suspended pivot and the pressure plate moves away from the flywheel by the knife edges, thereby compressing the clutch springs. This action removes the pressure from the clutch plate and thus moves back from the flywheel and the driven shaft becomes stationary. On the other hand, when the foot is taken off from the clutch pedal, the thrust bearing moves back by the levers. This allows the springs to extend and thus the pressure plate pushes the clutch plate back towards the flywheel.




The axial pressure exerted by the spring provides a frictional force in the circumferential direction when the relative motion between the driving and driven members tends to take place. If the torque due to this frictional force exceeds the torque to be transmitted, then no slipping takes place and the power is transmitted from the driving shaft to the driven shaft.

Now consider two friction surfaces, maintained in contact by an axial thrust $W$ , as shown in Fig. ($a$).

$T$ = Torque transmitted by the clutch

$p$ = Intensity of axial pressure with which the contact surfaces are held together,

$r_1$ and $r_2$ = External and internal radii of friction faces, and

∝ = Coefficient of friction.

Consider an elementary ring of radius $r$ and thickness $dr$ as shown in Fig. ($b$).

We know that area of contact surface or friction surface,

$= 2 \square r.dr$

Normal or axial force on the ring,

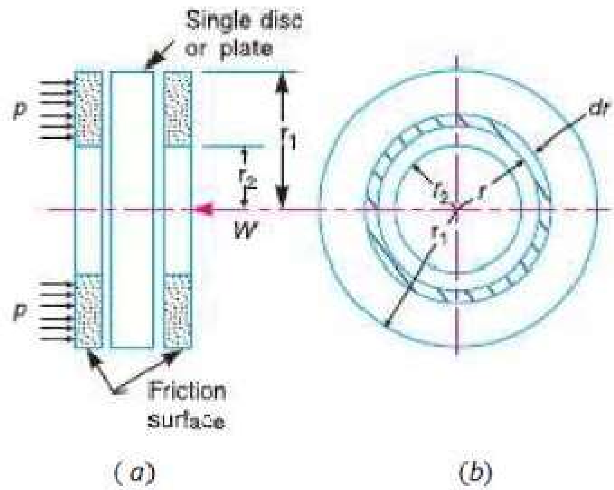
™$W$ = Pressure × Area = $p \times 2 \square r.dr$

and the frictional force on the ring acting tangentially at radius $r$,

$F_r = \propto.™W = \propto.p \times 2 \square r.dr$

Frictional torque acting on the ring,

$T_r = F_r \times r = \propto.p \times 2 \square r.dr \times r = 2 \square \times \propto .p.r_2\, dr$




We shall now consider the following two cases :

**1.** When there is a uniform pressure, and

**2.** When there is a uniform wear.

**1. *Considering uniform pressure***

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity of pressure,

$$p = \frac{W}{\square[(r_1)^2 \rceil (r_2)^2]}$$

We have discussed above that the frictional torque on the elementary ring of radius $r$ and thickness $dr$ is

Integrating this equation within the limits from $r_2$ to $r_1$ for the total frictional torque.

4 Total frictional torque acting on

$$T = \int_{r_1}^{r_2} 2 \square \propto p.r^2 .dr$$

Substituting the value of $p$ from e

$$T = 2 \square \propto \cdot \frac{}{\square[(r)}$$

$T_r = 2 \square \propto .p.r_2\, dr$

$$= \frac{2}{3} \cdot \propto .W \left| \frac{(r}{\square (} \right.$$

$R$ = Mean radiu

$$= \frac{2}{3} \left| \frac{(r_1)^3 \rceil}{\square (r_1)^2 \rceil} \right.$$

**2. *Considering uniform wear***

In Fig. 10.22, let $p$ be the normal intensity of pressure at a distance $r$ from the axis of the clutch. Since the intensity of pressure varies inversely with the distance, therefore

$$p.r = C \text{ (a constant) or } p = C/r \qquad \text{...}(i)$$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \cdot 2\pi C.dr = 2\pi C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\,dr = 2\pi C[r]_{r_2}^{r_1} = 2\pi C(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi(r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi\mu.\,p\,r^2.dr = 2\pi\mu \cdot \frac{C}{r}\,r^2.dr = 2\pi\mu.C.r.dr \qquad \text{...}(\because p = C/r)$$

∴ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C[(r_1)^2 - (r_2)^2] = \pi\mu \cdot \frac{W}{2\pi(r_1 - r_2)}[(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \cdot \mu.W(r_1 + r_2) = \mu.W.R$$

where $R$ = Mean radius of the friction surface = $\frac{r_1 + r_2}{2}$

**Multiple Disc Clutch**

A multiple disc clutch, as shown in Fig. 10.23, may be used when a large torque is to be transmitted. The inside discs (usually of steel) are fastened to the driven shaft to permit axial motion (except for the last disc). The outside discs (usually of bronze) are held by bolts and are fastened to the housing which is keyed to the driving shaft. The multiple disc clutches are extensively used in motor cars, machine tools etc.

*Let*

$n_1$ = Number of discs on the driving shaft, and

$n_2$ = Number of discs on the driven shaft.

4 Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

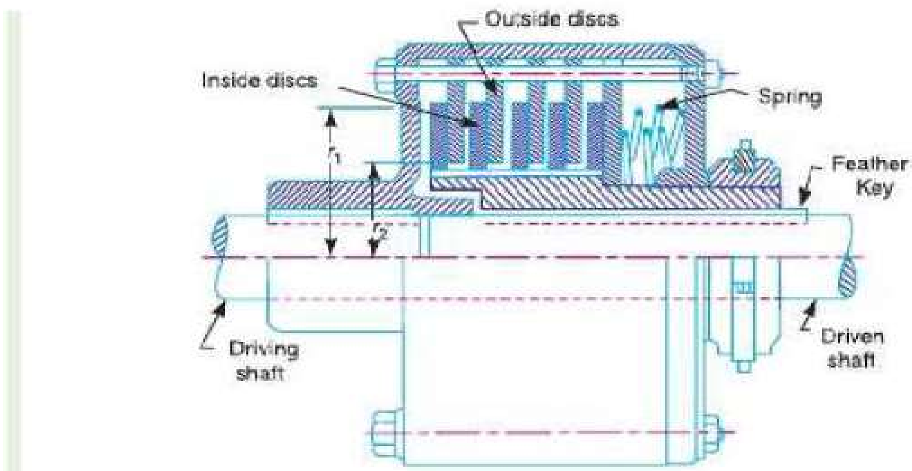
and total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

where $R$ = Mean radius of the friction surfaces

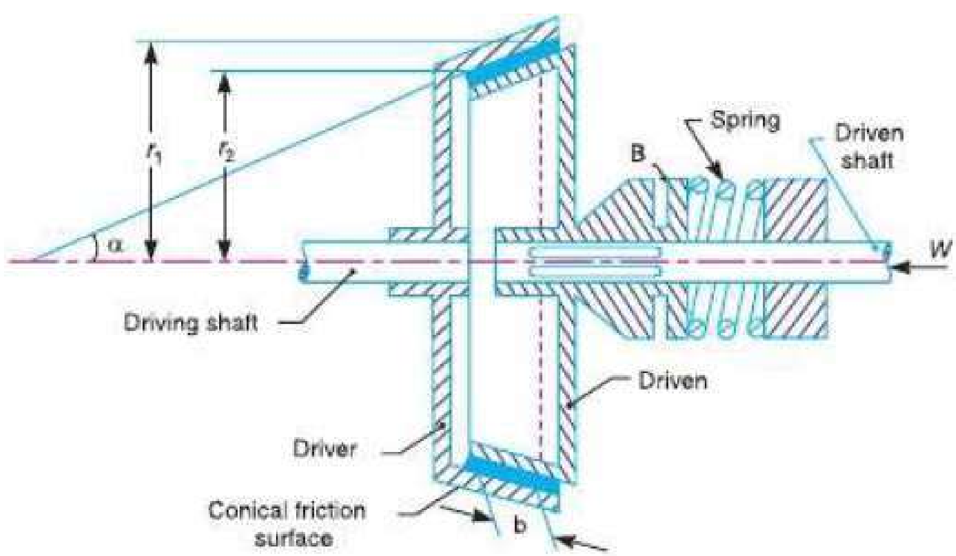
$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \quad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \quad \text{...(For uniform wear)}$$

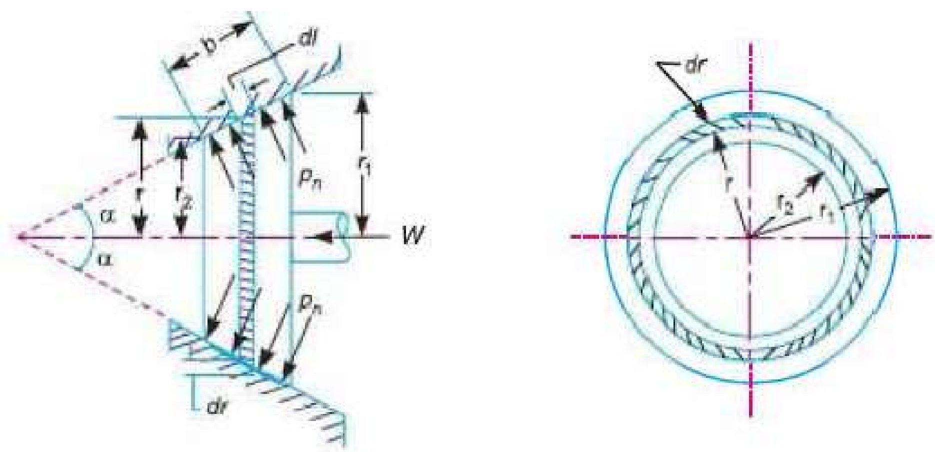



## Cone Clutch

A cone clutch, as shown in Fig. 10.24, was extensively used in automobiles but now-a-days it has been replaced completely by the disc clutch



It consists of one pair of friction surface only. In a cone clutch, the driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven. The driven member resting on the feather key in the driven shaft, may be shifted along the shaft by a forked lever provided at *B*, in order to engage the clutch by bringing the two conical surfaces in contact. Due to the frictional resistance set up at this contact surface, the torque is transmitted from one shaft to another. In some cases, a spring is placed around the driven shaft in contact with the hub of the driven. This spring holds the clutch faces in contact and maintains the pressure between them, and the forked lever is used only for disengagement of the clutch. The contact surfaces of the clutch may be metal to metal contact, but more often the driven member is lined with some material like wood, leather, cork or asbestos etc. The material of the clutch faces (*i.e.* contact surfaces) depends upon the allowable normal pressure and the coefficient of friction. Consider a pair of friction surface as shown in Fig. 10.25 (*a*). Since the area of contact of a pair of friction surface is a frustrum of a cone, therefore the torque transmitted by the cone clutch may be determined in the similar manner as discussed for conical pivot bearings in Art.




(a) (b)

$p_n$ = Intensity of pressure with which the conical friction surfaces are held together (*i.e.* normal pressure between contact surfaces),

$r_1$ and $r_2$ = Outer and inner radius of friction surfaces respectively.

$R$ = Mean radius of the friction surface

⟨ = Semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch,

∝ = Coefficient of friction between contact surfaces, and

$b$ = Width of the contact surfaces (also known as face width or clutch face).

Consider a small ring of radius $r$ and thickness $dr$, as shown in Fig. 10.25 ($b$). Let $dl$ is length of ring of the friction surface, such that

$$dl = dr.\text{cosec}\,\alpha$$

Area of the ring,

$$A = 2\pi\, r.dl = 2\pi r.dr\ \text{cosec}\,\alpha$$

We shall consider the following two cases :

**1.** When there is a uniform pressure, and

**2.** When there is a uniform wear.

***1. Considering uniform pressure***

We know that normal load acting on the ring,

$$W_n = \text{Normal pressure} \times \text{Area of ring} = p_n \times 2\pi\, r.dr.\text{cosec}\,\alpha$$

and the axial load acting on the ring,

$$W = \text{Horizontal component of } \delta W_n \ (i.e. \text{ in the direction of } W)$$

$$= \delta W_n \times \sin\alpha = p_n \times 2\pi\, r.dr.\ \text{cosec}\,\alpha \times \sin\alpha = 2\pi \times p_n.r.dr$$

Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\pi\, p_n r.dr = 2\pi p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\, p_n \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\, p_n \left[(r_1)^2 - (r_2)^2\right]$$

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$


$$W = \int_{r_2}^{r_1} 2\pi\, p_n r.dr = 2\pi p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\, p_n \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\, p_n \left[(r_1)^2 - (r_2)^2\right]$$

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$


We know that frictional force on the ring acting tangentially at radius $r$,

$$F_r = \mu.\delta W_n = \mu.p_n \times 2\pi\, r.dr.\text{cosec}\,\alpha$$

Frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p_n \times 2\pi\, r.dr.\ \text{cosec}\,\alpha.r = 2\pi\, \mu.p_n.\text{cosec}\,\alpha.r^2\, dr$$

Integrating this expression within the limits from $r_2$ to $r_1$ for the total frictional torque on the clutch.

Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu . p_n .\text{cosec}\,\alpha . r^2 .dr = 2\pi\mu\, p_n .\text{cosec}\,\alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu\, p_n .\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of $p_n$ from equation ***(i)***, we get

$$T = 2\pi\mu \times \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \times \text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \times \mu . W .\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

***2. Considering uniform wear***

In Fig. 10.25, let $p_r$ be the normal intensity of pressure at a distance $r$ from the axis of the clutch. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$\therefore \quad p_r . r = C$ (a constant) or $\quad p_r = C / r$

We know that the normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_r \times 2\pi r.dr\,\text{cosec}\,\alpha$$

and the axial load acting on the ring,

$$\delta W = \delta W_n \times \sin\alpha = p_r . 2\pi r.dr.\text{cosec}\,\alpha . \sin\alpha = p_r \times 2\pi r.dr$$

$$= \frac{C}{r} \times 2\pi r.dr = 2\pi C.dr \qquad \ldots(\because p_r = C / r)$$

$\therefore$ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C\, [r]_{r_2}^{r_1} = 2\pi C\,(r_1 - r_2)$$

or $$C = \frac{W}{2\pi(r_1 - r_2)} \qquad \ldots(iii)$$

We know that frictional force acting on the ring,

$$F_r = \mu . \delta W_n = \mu . p_r \times 2\pi r \times dr\,\text{cosec}\,\alpha$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = \mu\, p_r \times 2\pi r.dr.\text{cosec}\,\alpha \times r$$

$$= \mu \frac{C}{r} \times 2\pi r^2 .dr.\text{cosec}\,\alpha = 2\pi\mu . C\,\text{cosec}\,\alpha \times r\, dr$$

4 Total frictional torque acting on the clutch,

$$T = \int_{r_2}^{r_1} 2\pi\mu . C . \text{cosec}\,\alpha . r\, dr = 2\pi\mu . C . \text{cosec}\,\alpha \left[ \frac{r}{2} \right]_{r_2}^{r_1}$$

$$= 2\pi\mu . C . \text{cosec}\,\alpha \left[ \frac{(r_1)^2 - (r_2)^2}{2} \right]$$

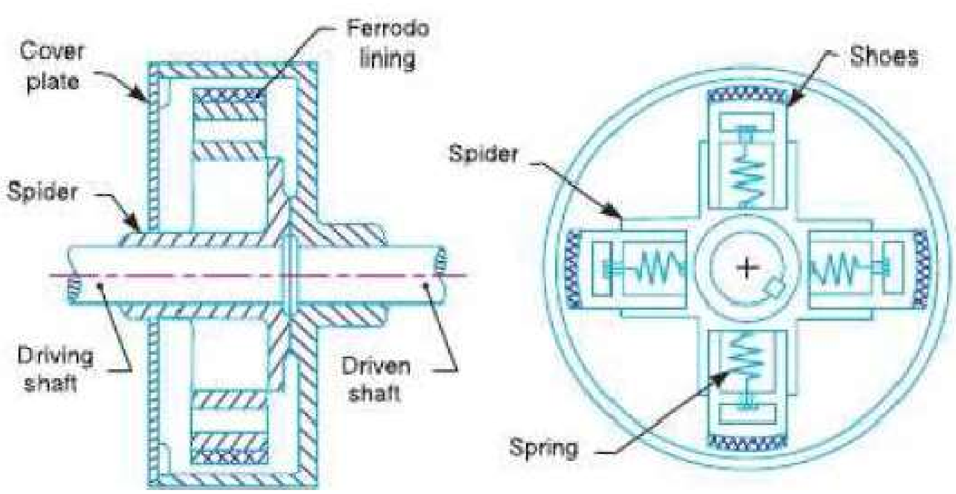
Substituting the value of $C$ from equation (*i*), we have

$$T = 2\pi\mu \cdot \frac{W}{2\pi(r_1 - r_2)} \cdot \text{cosec}\,\alpha \left[ \frac{(r_1)^2 - (r_2)^2}{2} \right]$$

$$= \mu . W \text{cosec}\,\alpha \left( \frac{r_1 + r_2}{2} \right) = \mu . W . R \text{ cosec}\,\alpha$$

**Centrifugal Clutch**

The centrifugal clutches are usually incorporated into the motor pulleys. It consists of a number of shoes on the inside of a rim of the pulley, as shown in Fig. 10.28. The outer surface of the shoes are covered with a friction material. These shoes, which can move radially in guides, are held




against the boss (or spider) on the driving shaft by means of springs. The springs exert a radially inward force which is assumed constant. The mass of the shoe, when revolving, causes it to exert a radially outward force (*i.e.* centrifugal force). The magnitude of this centrifugal force depends upon the speed at which the shoe is revolving. A little consideration will show that when the centrifugal force is less than the spring force, the shoe remains in the same position as when the driving shaft was stationary, but when the centrifugal force is equal to the spring force, the shoe is just floating. When the centrifugal force exceeds the spring force, the shoe moves outward and comes into contact with the driven member and presses against it. The force with which the shoe presses against the driven member is the difference of the centrifugal force and the spring force. The increase of speed causes the shoe to press harder

and enables more torque to be transmitted.

In order to determine the mass and size of the shoes, the following procedure is adopted :

***1. Mass of the shoes***

Consider one shoe of a centrifugal clutch as shown in Fig

Let $m$ = Mass of each shoe,

$n$ = Number of shoes,

$r$ = Distance of centre of gravity of the shoe from the centre of the spider,

$R$ = Inside radius of the pulley rim,

$N$ = Running speed of the pulley in r.p.m.,

$\omega$ = Angular running speed of the pulley in rad/s = $2\pi N/60$ rad/s,

$\omega_1$ = Angular speed at which the engagement begins to take place, and

$\propto$ = Coefficient of friction between the shoe and rim.

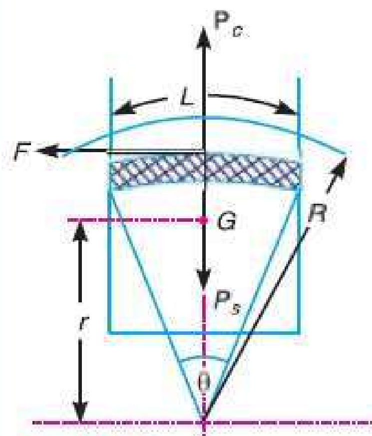



Fig. 10.29. Forces on a shoe of centrifugal clutch.

We know that the centrifugal force acting on each shoe at the running speed,

$$*P_c = m.\omega^2.r$$

We know that the centrifugal force acting on each shoe at the running speed,

$$*P_c = m.\omega^2.r$$

and the inward force on each shoe exerted by the spring at the speed at which engagement begins to take place,

$$P_s = m\,(\omega_1)^2\,r$$

The net outward radial force (*i.e.* centrifugal force) with which the shoe presses against the rim at the running speed

$$= P_c - P_s$$

and the frictional force acting tangentially on each shoe,

$$F = \propto (P_c - P_s)$$

Frictional torque acting on each shoe,

$$= F \times R = \propto (P_c - P_s)\,R$$

and total frictional torque transmitted,

$$T = \propto (P_c - P_s)\, R \times n = n.F.R$$

From this expression, the mass of the shoes ($m$) may be evaluated.

***2. Size of the shoes***

$l$ = Contact length of the shoes,

$b$ = Width of the shoes,

$R$ = Contact radius of the shoes. It is same as the inside radius of the rim of the pulley.

$\theta$ = Angle subtended by the shoes at the centre of the spider in radians.

$p$ = Intensity of pressure exerted on the shoe. In order to ensure reasonable life, the intensity of pressure may be taken as 0.1 $N/mm^2$

Area of contact of the shoe,

$$A = l.b$$

and the force with which the shoe presses against the rim

$$= A \times p = l.b.p$$

Since the force with which the shoe presses against the rim at the running speed is $(P_c - P_s)$, therefore

$$l.b.p = P_c - P_s$$

From this expression, the width of shoe ($b$) may be obtained.

## Introduction

A ***brake*** is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators etc. The energy absorbed by brakes is dissipated in the form of heat. This heat is dissipated in the surrounding air (or water which is circulated through the passages in the brake drum) so that excessive heating of the brake lining does not take place. The capacity of a brake depends upon the following factors :

**1.** The unit pressure between the braking surfaces,

**2.** The coefficient of friction between the braking surfaces,

**3.** The peripheral velocity of the brake drum,

**4.** The projected area of the friction surfaces, and

**5.** The ability of the brake to dissipate heat equivalent to the energy being absorbed.

The major functional difference between a clutch and a brake is that a clutch is used to keep the driving and driven member moving together, whereas brakes are used to stop a moving member or to control its speed.

## Materials for Brake Lining

The material used for the brake lining should have the following characteristics

**1.** It should have high coefficient of friction with minimum fading. In other words, the coeffi- cient of friction should remain constant with change in temperature.

**2.** It should have low wear rate.

**3.** It should have high heat resistance.

**4.** It should have high heat dissipation capacity.

**5.** It should have adequate mechanical strength.

**6.** It should not be affected by moisture and oil.

The materials commonly used for facing or lining of brakes and their properties are shown in the following table.

## Types of Brakes

The brakes, according to the means used for transforming the energy by the braking elements, are classified as :

**1.** Hydraulic brakes *e.g*. pumps or hydrodynamic brake and fluid agitator,

**2.** Electric brakes *e.g*. generators and eddy current brakes, and

**3.** Mechanical brakes.

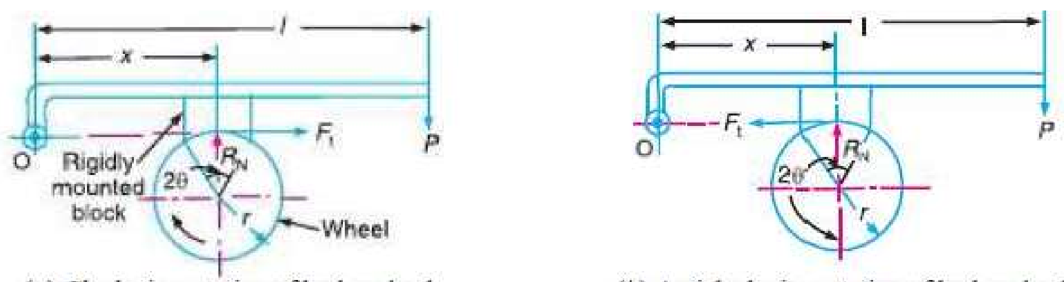
The hydraulic and electric brakes cannot bring the member to rest and are mostly used where large amounts of energy are to be transformed while the brake is retarding the load such as in laboratory dynamometers, high way trucks and electric locomotives. These brakes are also used for retarding or controlling the speed of a vehicle for down-hill travel. The mechanical brakes, according to the direction of acting force, may be divided into the following two groups :

**(a)** ***Radial brakes.*** In these brakes, the force acting on the brake drum is in radial direction. The radial brakes may be sub-divided into ***external brakes*** and ***internal brakes.*** According to the shape of the friction elements, these brakes may be ***block*** or ***shoe brakes*** and ***band brakes.***

**(b)** ***Axial brakes.*** In these brakes, the force acting on the brake drum is in axial direction. The axial brakes may be disc brakes and cone brakes. The analysis of these brakes is similar to clutches. Since we are concerned with only mechanical brakes, therefore, these are discussed, in detail, in the following pages.

## Single Block or Shoe Brake

A single block or shoe brake is shown in Fig. 19.1. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum $O$.




(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on the wheel,

$$F_t = \mu . R_N$$

and the braking torque, $T_B = F_t . r = \mu . R_N . r$

Let us now consider the following three cases :

**Case 1.** When the line of action of tangential braking force ($F_t$ ) passes through the fulcrum $O$ of the lever, and the brake wheel rotates clockwise as shown in Fig. (*a*), then for equilibrium, taking moments about the fulcrum $O$, we have

$$R_N \cdot x = P \cdot l \text{ or } R_N = \frac{P \cdot l}{x}$$

Braking torque,

$$T_B = \propto R_N \cdot r = \propto \cdot \frac{P.l}{x} \cdot r = \frac{\propto .P.l.r}{x}$$

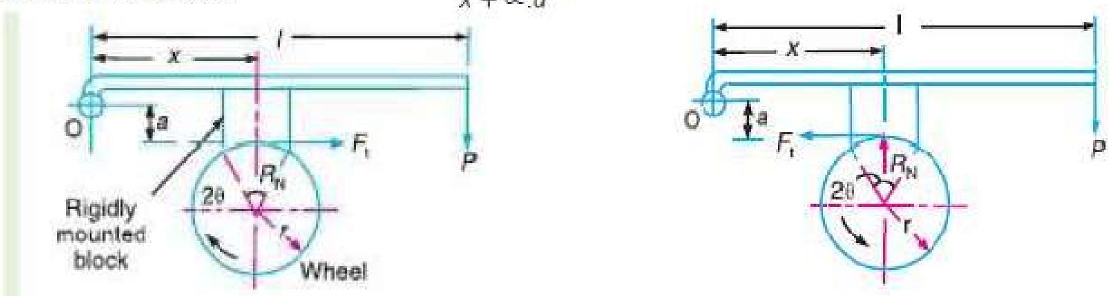
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. (*b*), then the braking torque is same, *i.e*

$$T_B = \propto .R_N .r = \frac{\propto .P.l.r}{x}$$

**Case 2.** When the line of action of the tangential braking force (*Ft* ) passes through a distance '*a*' below the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. (*a*), then for equilibrium, taking moments about the fulcrum *O*,

$$R_N \times x + F_t \times a = P.l \quad \text{or } R_N \times x + \mu R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \propto .a}$$

and braking torque, $T_B = \propto R_N \ r = \dfrac{\propto . p.l.r}{x + \propto .a}$




(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 (*b*), then for equilibrium.

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a$$

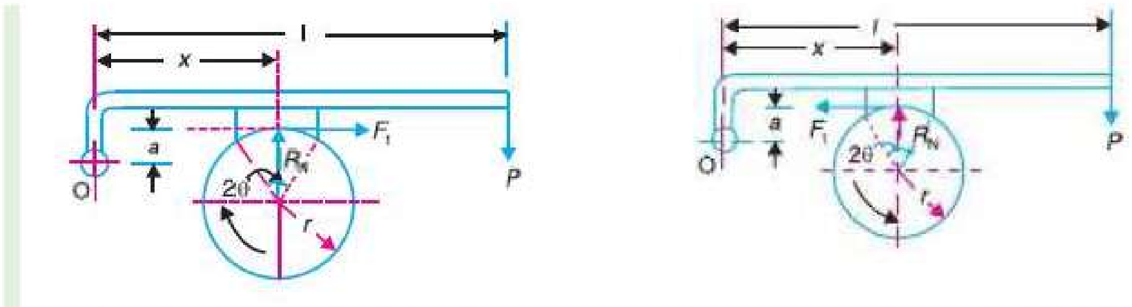
or $\quad R_N (x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \propto .a}$

and braking torque, $\quad T_B = \propto .R_N .r = \dfrac{\propto .P.l.r}{x - \propto .a}$

**Case 3.** When the line of action of the tangential braking force (*Ft* ) passes through a distance '*a*' above the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. 19.3 (*a*), then for equilibrium, taking moments about the fulcrum *O*, we have

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a$$

or $\quad R_N(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \mu.a}$




(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

and braking torque,

$$T_B = \mu.R_N.r = \frac{\mu.P.l.r}{x - \mu.a}$$

When the brake wheel rotates anticlockwise as shown in Fig. 19.3 (*b*), then for equilibrium, taking moments about the fulcrum *O*, we have

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu.R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \mu.a}$$

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\mu.P.l.r}{x + \mu.a}$

### Pivoted Block or Shoe Brake

We have discussed in the previous article that when the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. But when the angle of contact is greater than 60°, then the unit pressure normal to the surface of contact is less at the ends than at the centre. In such cases, the block or shoe is pivoted to the lever, as shown in Fig. 19.4, instead of being rigidly attached to the lever. This gives uniform wear of the brake lining in the direction of the applied force. The braking torque for a pivoted block or shoe brake (*i.e.* when $2\theta > 60°$) is

given by

$$T_B = F_t \cdot r = \mu'.R_N.r$$

where $\quad \mu'$ = Equivalent coefficient of friction = $\dfrac{4\mu \sin\theta}{2\theta + \sin 2\theta}$, and

$\mu$ = Actual coefficient of friction.

These brakes have more life and may provide a higher braking torque.

## Simple Band Brake

A band brake consists of a flexible band of leather, one or more ropes,or a steel lined with friction material, which embraces a part of the circumference of the drum. A band brake, as shown in Fig., is called a ***simple band brake*** in which one end of the band is attached to a fixed pin or fulcrum of the lever while the other end is attached to the lever at a distance *b* from the fulcrum. When a force *P* is applied to the lever at *C*, the lever turns about the fulcrum pin *O* and tightens the band on the drum and hence the brakes are applied. The friction between the band and the drum provides the braking force. The force *P* on the lever at *C* may be determined as discussed below :

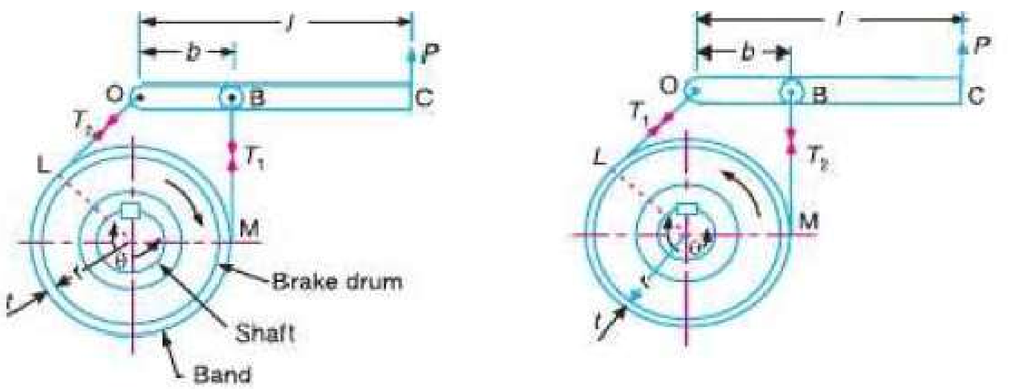
$\theta$ = Angle of lap (or embrace) of the band on the drum,

$\mu$ = Coefficient of friction between the band and the drum,

$r$ = Radius of the drum,

$t$ = Thickness of the band, and

$r_e$ = Effective radius of the drum




(*a*) Clockwise rotation of drum. (*b*) Anticlockwise rotation of drum.

We know that limiting ratio of the tensions is given by the relation,

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{or} \quad 2.3\log\left(\frac{T_1}{T_2}\right) = \mu.\theta$$

and braking force on the drum = $T_1 - T_2$

Braking torque on the drum,

$T_B = (T_1 - T_2)\, r$ . . . (Neglecting thickness of band)

$= (T_1 - T_2)\, r_e$ . . . (Considering thickness of band)

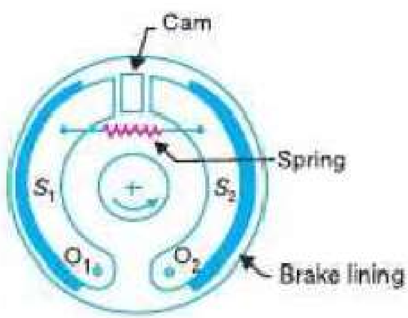
Now considering the equilibrium of the lever *OBC*. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig.(*a*), the end of the band attached to the fulcrum *O* will be slack with tension $T_2$ and end of the band attached to *B* will be tight with tension $T_1$. On the other hand, when the drum rotates in the anticlockwise direction, as shown in Fig.(*b*), the tensions in the band will reverse, *i.e.* the end of the band attached to the fulcrum *O* will be tight with tension $T_1$ and the end of the band attached to *B* will be slack with tension $T_2$. Now taking moments about the fulcrum *O*, we have

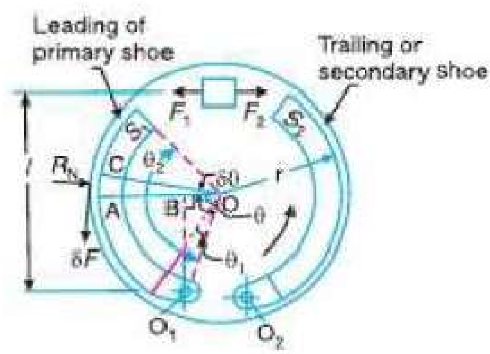
$P.l = T_1.b$ . . . (For clockwise rotation of the drum)

$P.l = T_2.b$ . . . (For anticlockwise rotation of the drum)

## Internal Expanding Brake

An internal expanding brake consists of two shoes $S_1$ and $S_2$ as shown in Fig.. The outer surface of the shoes are lined with some friction material (usually with Ferodo) to increase the coefficient of friction and to prevent wearing away of the metal. Each shoe is pivoted at one end about a fixed fulcrum $O_1$ and $O_2$ and made to contact a cam at the other end. When the cam rotates, the shoes are pushed outwards against the rim of the drum. The friction between the shoes and the drum produces the braking torque and hence reduces the speed of the drum. The shoes are




normally held in off position by a spring as shown in Fig. 19.24. The drum encloses the entire mechanism to keep out dust and moisture. This type of brake is commonly used in motor cars and light trucks.

We shall now consider the forces acting on such a brake, when the drum rotates in the anticlockwise direction as shown in Fig. 19.25. It may be noted that for the anticlockwise direction, the left hand shoe is known as ***leading*** or ***primary shoe*** while the right hand shoe is known as ***trailing*** or ***secondary shoe.***

Let $r$ = Internal radius of the wheel rim,

$b$ = Width of the brake lining,

$p_1$ = Maximum intensity of normal pressure,

$p_N$ = Normal pressure,

$F_1$ = Force exerted by the cam on the leading shoe, and

$F_2$ = Force exerted by the cam on the trailing shoe.

Consider a small element of the brake lining $AC$ subtending an angle $\delta\theta$ at the centre. Let $OA$ makes an angle $\theta$ with $OO_1$ as shown in Fig. 19.25. It is assumed that the pressure distribution on the shoe is nearly uniform, however the friction lining wears out more at the free end. Since the shoe turns about $O_1$, therefore the rate of wear of the shoe lining at $A$ will be proportional to the radial displacement of that point. The rate of wear of the shoe lining varies directly as the perpendicular distance from $O_1$ to $OA$, *i.e.* $O_1B$. From the geometry of the figure,

$$O_1B = OO_1 \sin\theta$$

and normal pressureat $A$,

$$p_N \quad \sin\theta \text{ or } p_N = p_1 \sin\theta$$

Normal force acting on the element,

$$\delta R_N = \text{Normal pressure} \times \text{Area of the element}$$

$$= p_N (b.r.\delta\theta) = p_1 \sin\theta (b.r.\delta\theta)$$

and braking or friction force on the element,

$$\delta F = \mu \cdot \delta R_N = \mu . p_1 \sin\theta (b.r.\delta\theta)$$

∴ Braking torque due to the element about $O$,

$$\delta T_B = \delta F \cdot r = \mu . p_1 \sin\theta (b.r.\delta\theta) r = \mu . p_1 b\, r^2 (\sin\theta\, \delta\theta)$$

and total braking torque about $O$ for whole of one shoe,

$$T_B = \mu\, p_1 b\, r^2 \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \mu\, p_1 b\, r^2 \left[-\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b r^2 (\cos\theta_1 - \cos\theta_2)$$

Moment of normal force $\delta R_N$ of the element about the fulcrum $O_1$,

$$\delta M_N = \delta R_N \cdot O_1B = \delta R_N (OO_1 \sin\theta)$$

$$= p_1 \sin\theta (b.r.\delta\theta)(OO_1 \sin\theta) = p_1 \sin^2\theta (b.r.\delta\theta) OO_1$$

∴ Total moment of normal forces about the fulcrum $O_1$,

$$= p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \frac{1}{2}(1 - \cos 2\theta) d\theta \qquad \ldots \left[\because \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \theta_1 + \frac{\sin 2\theta_1}{2}\right]$$

$$= \frac{1}{2} p_1.b.r.OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

Moment of frictional force $\delta F$ about the fulcrum $O_1$,

$$\delta M_F = \delta F \cdot AB = \delta F\,(r - OO_1 \cos\theta) \qquad \ldots (\because AB = r - OO_1\cos\theta)$$

$$= \mu\, p_1 \sin\theta\,(b.r.\delta\theta)\,(r - OO_1\cos\theta)$$

$$= \mu.\, p_1.b.r\,(r\sin\theta - OO_1 \sin\theta\cos\theta)\,\delta\theta$$

$$= \mu.\, p_1.b.r\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right)\delta\theta \qquad \ldots (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

∴ Total moment of frictional force about the fulcrum $O_1$,

$$M_F = \mu\, p_1 b\, r \int_{\theta_1}^{\theta_2}\left(r\sin\theta - \frac{OO_1}{2}\sin 2\theta\right) d\theta$$

$$= \mu\, p_1 b\, r\left[-r\cos\theta + \frac{OO_1}{4}\cos 2\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b\, r\left[-r\cos\theta_2 + \frac{OO_1}{4}\cos 2\theta_2 + r\cos\theta_1 - \frac{OO_1}{4}\cos 2\theta_1\right]$$

$$= \mu\, p_1 b\, r\left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4}(\cos 2\theta_2 - \cos 2\theta_1)\right]$$

Now for leading shoe, taking moments about the fulcrum $O_1$,

$$F_1 \times l = M_N - M_F$$

and for trailing shoe, taking moments about the fulcrum $O_2$,

$$F_2 \times l = M_N + M_F$$

## Types of Dynamometers

Following are the two types of dynamometers, used for measuring the brake power of an engine.

**1.** Absorption dynamometers, and

**2.** Transmission dynamometers.

In the *absorption* ***dynamometers,*** the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the ***transmission dynamometers,*** the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitably measured.

## Classification of Absorption Dynamometers

The following two types of absorption dynamometers are important from the subject point of view :

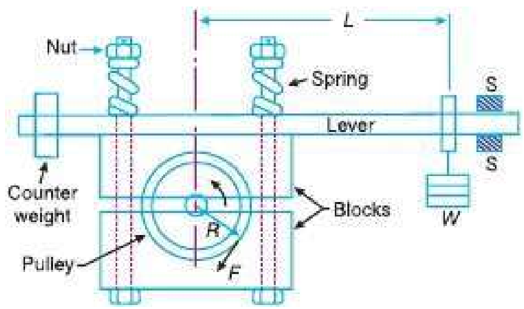
**1.** Prony brake dynamometer, and

**2.** Rope brake dynamometer.

These dynamometers are discussed, in detail, in the following pages.

## Prony Brake Dynamometer

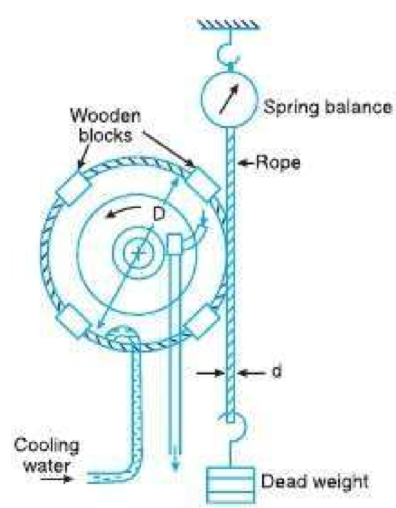
A simplest form of an absorption type dynamometer is a prony brake dynamometer, as shown in Fig. 19.31. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig.. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight $W$ at its outer end. A counter weight is placed at the other end of the lever which balances the brake when unloaded. Two stops S, S are provided to limit the motion of the lever




When the brake is to be put in operation, the long end of the lever is loaded with suitable weights $W$ and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions, the moment due to the weight $W$ must balance the mo- ment of the frictional resistance between the blocks and the pulley.

## Rope Brake Dynamometer

It is another form of absorption type dynamometer which is most commonly used for measur- ing the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig.. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel. In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.



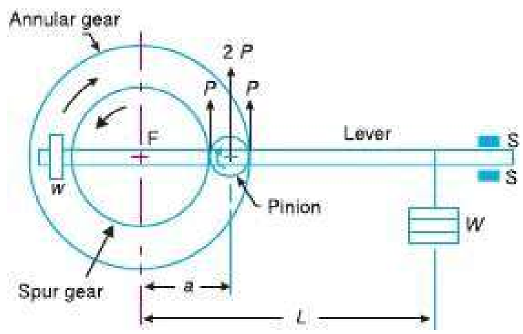
## Classification of Transmission Dynamometers

The following types of transmission dynamometers are important from the subject point of view :

1. Epicyclic-train dynamometer,
2. Belt transmission dynamometer, and
3. Torsion dynamometer.

We shall now discuss these dynamometers, in detail, in the following pages.

## Epicyclic-train Dynamometer




An epicyclic-train dynamometer, as shown in Fig. 19.33, consists of a simple epicyclic train of gears, *i.e.* a spur gear, an annular gear (a gear having internal teeth) and a pinion. The spur gear is keyed to the engine shaft (*i.e.* driving shaft) and rotates in anticlockwise direction. The annular gear is also keyed to the driving shaft and rotates in clockwise direction. The pinion or the intermediate gear meshes with both the spur and annular gears. The pinion revolves freely on a lever which is pivoted to the common axis of the driving and driven shafts. A weight $w$ is placed at the smaller end of the lever in order to keep it in position. A little consideration will show that if the friction of the pin on which the pinion rotates is neglected, then the tangential effort $P$ exerted by the spur gear on the pinion and the tangential reaction of the annular gear on the pinion are equal.

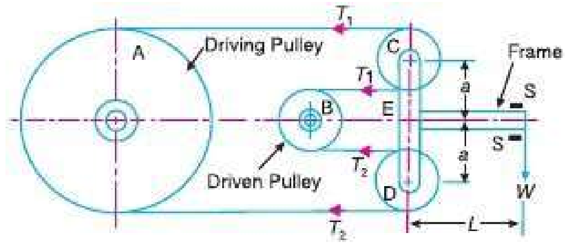
Since these efforts act in the upward direction as shown, therefore total upward force on the lever acting through the axis of the pinion is $2P$. This force tends to rotate the lever about its fulcrum and it is balanced by a dead weight $W$ at the end of the lever. The stops $S$, $S$ are provided to control the movement of the lever.

For equilibrium of the lever, taking moments about the fulcrum $F$,

$$2P \times a = W.L \quad \text{or} \quad P = W.L / 2a$$

## Belt Transmission Dynamometer-Froude or Throneycroft Transmission Dynamometer

When the belt is transmitting power from one pulley to another, the tangential effort on the driven pulley is equal to the difference between the tensions in the tight and slack sides of the belt. A belt dynamometer is introduced to measure directly the difference between the tensions of the belt, while it is running.




A belt transmission dynamometer, as shown in Fig. 19.34, is called a Froude or Throneycroft transmission dynamometer. It consists of a pulley *A* (called driving pulley) which is rigidly fixed to the shaft of an engine whose power is required to be measured. There is another pulley *B* (called driven pulley) mounted on another shaft to which the power from pulley *A* is transmitted. The pulleys *A* and *B* are connected by means of a continuous belt passing round the two loose pulleys *C* and *D* which are mounted on a *T*-shaped frame. The frame is pivoted at *E* and its movement is controlled by two stops *S*,*S*. Since the tension in the tight side of the belt ($T_1$) is greater than the tension in the slack side of the belt ($T_2$), therefore the total force acting on the pulley *C* (*i.e.* $2T_1$) is greater than the total force acting on the pulley *D* (*i.e.* $2T_2$). It is thus obvious that the frame causes movement about *E* in the anticlockwise direction. In order to balance it, a weight *W* is applied at a distance *L* from *E* on the frame as shown in Fig.

Now taking moments about the pivot *E*, neglecting friction,

$$2T_1 \cdot a = 2T_2 \cdot a + W .$$

## Torsion Dynamometer

A torsion dynamometer is used for measuring large powers particularly the power transmit- ted along the propeller shaft of a turbine or motor vessel. A little consideration will show that when the power is being transmitted, then the driving end of the shaft twists through a small angle relative to the driven end of the shaft. The amount of twist depends upon many factors such as torque acting on the shaft (*T*), length of the shaft (*l*), diameter of the shaft (*D*) and modulus of rigidity (*C*) of the material of the shaft. We know that the torsion equation is

## TURNING MOMENT DIAGRAM AND FLY WHEELS

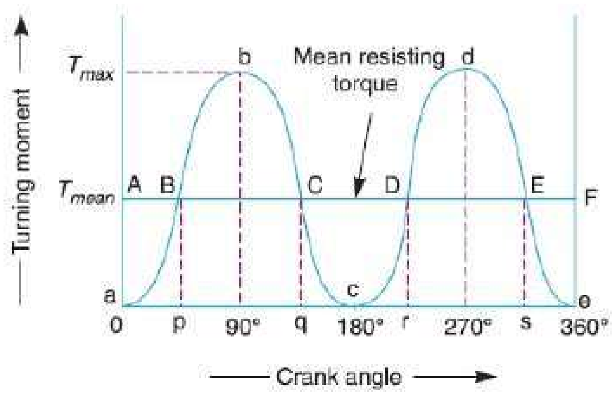
**Turning Moment Diagram:** The turning moment diagram is graphical representation of the turning moment or crank effort for various positions of crank.

**Single cylinder double acting engine:**

A turning moment diagram for a single cylinder double acting steam engine is shown in Fig. The vertical ordinate represents the turning moment and the horizontal ordinate represents the crank angle.

the turning moment on the crankshaft,

$$T = F_P \times r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right)$$




Turning moment diagram for a single cylinder, double acting steam engine.

where $F_P$ = Piston effort,

$r$ = Radius of crank,

$n$ = Ratio of the connecting rod length and radius of crank, and

$\theta$ = Angle turned by the crank from inner dead centre.

From the above expression, we see that the turning moment ($T$) is zero, when the crank angle ($\theta$) is zero. It is maximum when the crank angle is 90° and it is again zero when crank angle is 180°.

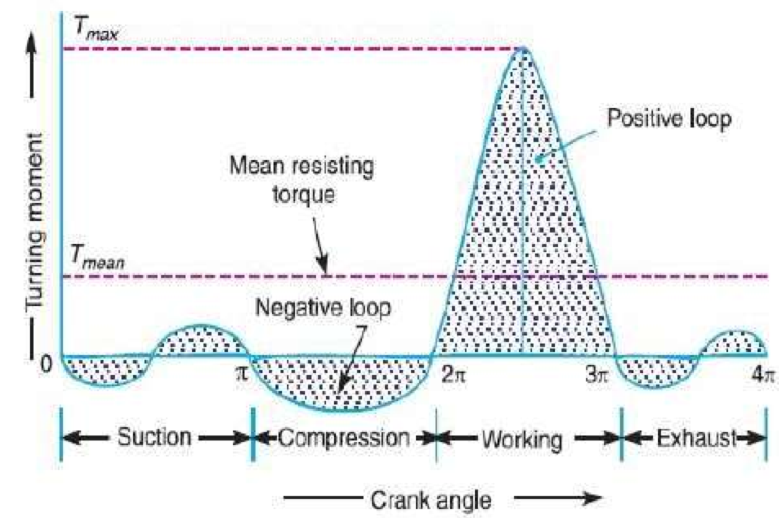
This is shown by the curve *abc* in Fig. and it represents the turning moment diagram for outstroke. The curve *cde* is the turning moment diagram for instroke and is somewhat similar to the curve *abc*.

Since the work done is the product of the turning moment and the angle turned, therefore the area of the turning moment diagram represents the work done per revolution. In actual practice, the engine is assumed to work against the mean resisting torque, as shown by a horizontal line *AF*. The height of the ordinate *aA* represents the mean height of the turning moment diagram. Since it is assumed that the work done by the turning moment per revolution is equal to the work done against the mean resisting torque, therefore the area of the rectangle *aAFe* is proportional to the work done against the mean resisting torque.


For flywheel, have a look at your tailor's manual sewing machine.

**Turning moment diagram for 4-stroke I.C engine:**

A turning moment diagram for a four stroke cycle internal combustion engine is shown in Fig. We know that in a four stroke cycle internal combustion engine, there is one working stroke after the crank has turned through two revolutions, *i.e.* 720° (or 4 π radians).

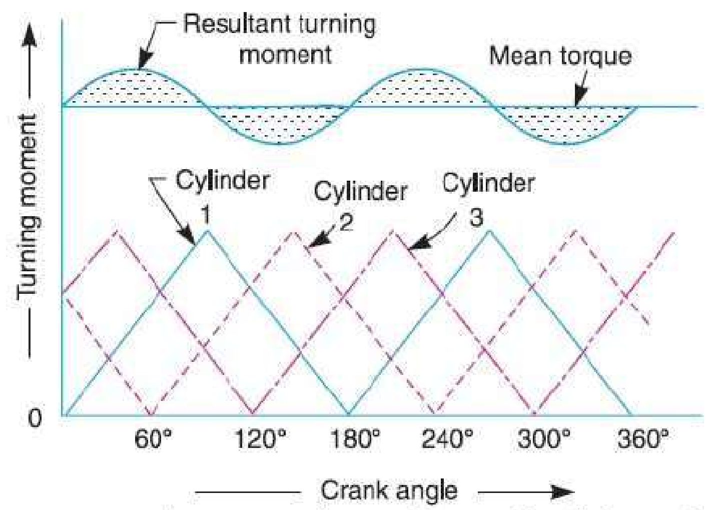



Turning moment diagram for a four stroke cycle internal combustion engine.

Since the pressure inside the engine cylinder is less than the atmospheric pressure during the suction stroke, therefore a negative loop is formed as shown in Fig. 16.2. During the compression stroke, the work is done on the gases, therefore a higher negative loop is obtained. During the expansion or working stroke, the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases, therefore a negative loop is formed. It may be noted that the effect of the inertia forces on the piston is taken into account in Fig.

**Turning moment diagram for a multi cylinder engine**:

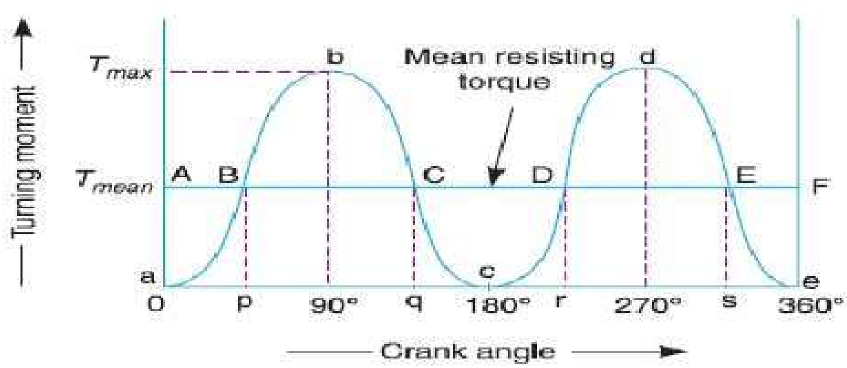
A separate turning moment diagram for a compound steam engine having three cylinders and the resultant turning moment diagram is shown in Fig. The resultant turning moment diagram is the sum of the turning moment diagrams for the three cylinders. It may be noted that the first cylinder is the high pressure cylinder, second cylinder is the intermediate cylinder and the third cylinder is the low pressure cylinder. The cranks, in case of three cylinders, are usually placed at 120° to each other.




Turning moment diagram for a multi-cylinder engine.

**Fluctuation of Energy:**

The difference in the kinetic energies at the point is called the maximum fluctuation of energy.



The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig. We see that the mean resisting torque line *AF* cuts the turning moment diagram at points *B*, *C*, *D* and *E*. When the crank moves from *a* to *p*, the work done by the engine is equal to the area *aBp*, whereas the energy required is represented by the area *aABp*. In other words, the engine has done less work (equal to the area *a AB*) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from *p* to *q*, the work done by the engine is equal to the area *pBbCq*, whereas the requirement of energy is represented by the area *pBCq*. Therefore, the engine has done more work than the requirement. This excess work (equal to the area *BbC*) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from *p* to *q*.

Similarly, when the crank moves from *q* to *r*, more work is taken from the engine than is developed. This loss of work is represented by the area *C c D*. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from *q* to *r*. As the crank moves from *r* to *s*, excess energy is again developed given by the area *D d E* and the speed again increases. As the piston moves from *s* to *e*, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called ***fluctuations of energy***. The areas *BbC*, *CcD*, *DdE*, etc. represent fluctuations of energy.

A little consideration will show that the engine has a maximum speed either at *q* or at *s*. This is due to the fact that the flywheel absorbs energy while the crank moves from *p* to *q* and from *r* to *s*. On the other hand, the engine has a minimum speed either at *p* or at *r*. The reason is that the flywheel gives out some of its energy when the crank moves from *a* to *p* and *q* to *r*. The difference between the maximum and the minimum energies is known as ***maximum fluctuation of energy***.

**Fluctuation of Speed**:

This is defined as the ratio of the difference between the maximum and minimum angular speeds during a cycle to the mean speed of rotation of the crank shaft.

**Maximum fluctuation of energy:**

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. The horizontal line *AG* represents the mean torque line. Let $a_1$, $a_3$, $a_5$ be the areas above the mean torque line and $a_2$, $a_4$ and $a_6$ be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine.

Let the energy in the flywheel at $A = E$, then from Fig. we have

Energy at $B = E + a_1$

Energy at $C = E + a_1 - a_2$

Energy at $D = E + a_1 - a_2 + a_3$

Energy at $E = E + a_1 - a_2 + a_3 - a_4$

Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5$

Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

= Energy at $A$ (*i.e.* cycle repeats after $G$)

A flywheel stores energy when the supply is in excess and releases energy when energy is in deficit.

Let us now suppose that the greatest of these energies is at $B$ and least at $E$. Therefore,

Maximum energy in flywheel

$$= E + a_1$$

Minimum energy in the flywheel

$$= E + a_1 - a_2 + a_3 - a_4$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4) = a_2 - a_3 + a_4$$

**Coefficient of fluctuation of energy:**

It may be defined as the **ratio of the maximum fluctuation of energy to the work done per cycle.** Mathematically, coefficient of fluctuation of energy,

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

The work done per cycle (in N-m or joules) may be obtained by using the following two relations :

**1.** Work done per cycle $= T_{mean} \times \theta$

where $T_{mean}$ = Mean torque, and

$\theta$ = Angle turned (in radians), in one revolution.

= $2\pi$, in case of steam engine and two stroke internal combustion engines

= $4\pi$, in case of four stroke internal combustion engines.

The mean torque ($T_{mean}$) in N-m may be obtained by using the following relation :

$$T_{mean} = \frac{P \times 60}{2\pi N} = \frac{P}{\omega}$$

where $P$ = Power transmitted in watts,

$N$ = Speed in r.p.m., and

$\omega$ = Angular speed in rad/s $= 2\pi N/60$

2. The work done per cycle may also be obtained by using the following relation :

$$\text{Work done per cycle} = \frac{P \times 60}{n}$$

where $n$ = Number of working strokes per minute,

= $N$, in case of steam engines and two stroke internal combustion engines,

= $N/2$, in case of four stroke internal combustion engines.

**Coefficient of fluctuation of speed:**

The difference between the maximum and minimum speeds during a cycle is called the *maximum fluctuation of speed*. The ratio of the maximum fluctuation of speed to the mean speed is called the *coefficient of fluctuation of speed*.

Let $N_1$ and $N_2$ = Maximum and minimum speeds in r.p.m. during the cycle, and

$$N = \text{Mean speed in r.p.m.} = \frac{N_1 + N_2}{2}$$

∴ Coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

$$= \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \quad \text{...(In terms of angular speeds)}$$

$$= \frac{v_1 - v_2}{v} = \frac{2(v_1 - v_2)}{v_1 + v_2} \quad \text{...(In terms of linear speeds)}$$

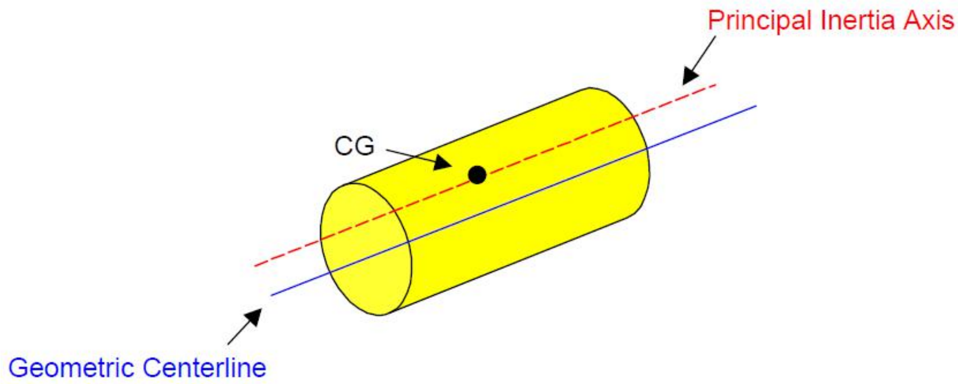
**Energy stored in flywheel**:

A flywheel is a rotating mass that is used as an energy reservoir in a machine. It absorbs energy in the form of kinetic energy, during those periods of crank rotation when actual turning moment is greater than the resisting moment and release energy, by way of parting with some of its K.E, when the actual turning moment is less than the resisting moment.

# MODULE-IV

DYNAMICS OF MACHINES

# BALANCING
# OF
# ROTATING MASSES




**Rotating centerline:**

The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

**Geometric centerline:**

The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:

**Static Unbalance** – where the PIA is displaced parallel to the geometric centerline. (Shown above)

**Couple Unbalance** – where the PIA intersects the geometric centerline at the center of gravity. (CG)

**Dynamic Unbalance** – where the PIA and the geometric centerline do not coincide or touch.

The most common of these is dynamic unbalance.

**Causes of Unbalance:**

In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. slight variation in the density of the material or
2. inaccuracies in the casting or
3. inaccuracies in machining of the parts.

**Why balancing is so important?**

1. A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2. As machines get bigger and go faster, the effect of the unbalance is much more severe.
3. The force caused by unbalance increases by the square of the speed.
4. If the speed is doubled, the force quadruples; if the speed is tripled the force increases

by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

## BALANCING:

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.
The objectives of balancing an engine are to ensure:

1. That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
2. That the couples involved in acceleration of the different moving parts balance each other.

### Types of balancing:

a) **Static Balancing:**
   i) Static balancing is a balance of forces due to action of gravity.
   ii) A body is said to be in static balance when its centre of gravity is in the axis of rotation.

b) **Dynamic balancing:**
   i) Dynamic balance is a balance due to the action of inertia forces.
   ii) A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.
   iii) The conditions of dynamic balance are met, the conditions of static balance are also met.

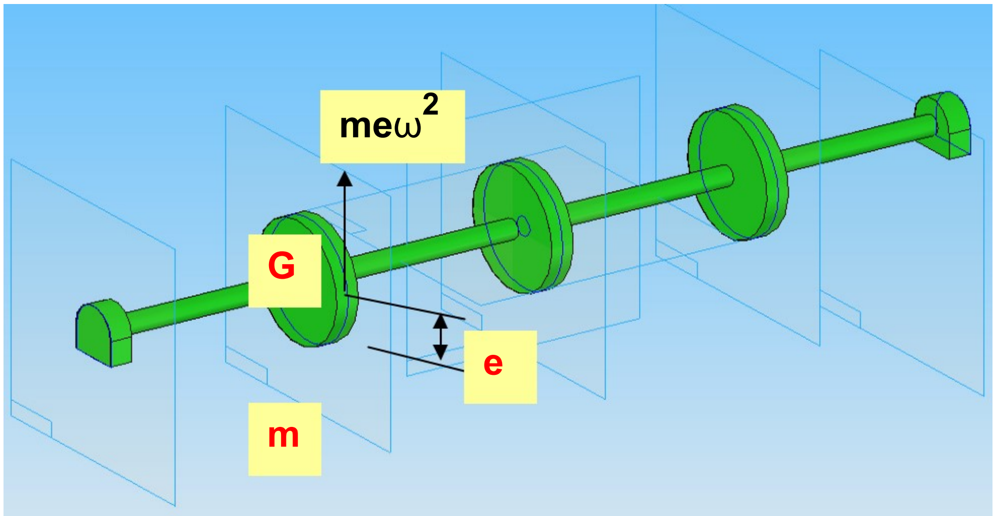
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

## BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.




The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members Balancing of rotating masses can be of

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of several masses rotating in the same plane
4. Balancing of several masses rotating in different planes

**STATIC BALANCING:**

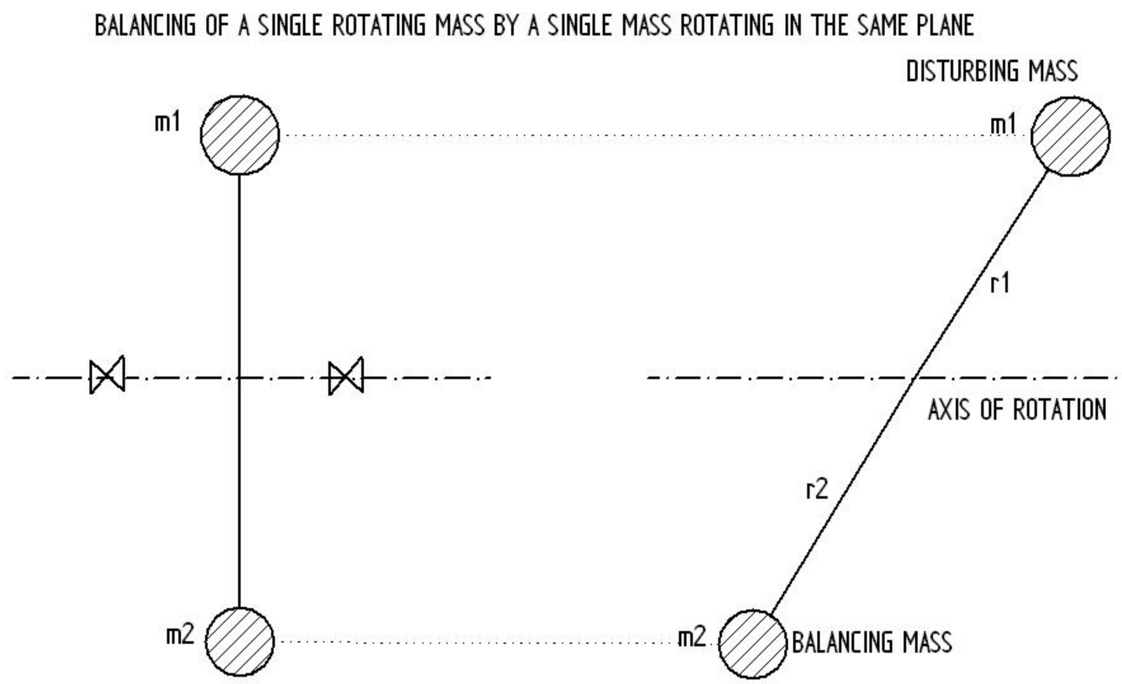
A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation

**DYNAMIC BALANCING;**

When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

**CASE 1.**

## BALANCING OF A SINGLE ROTATING MASS BY A SINGLE MASS ROTATING IN THE SAME PLANE




Consider a disturbing mass $m_1$ which is attached to a shaft rotating at ω rad/s.
Let

$r_1$ = radius of rotation of the mass $m_1$

= distancebetweenthe axis of rotationof the shaft and the centreof gravity of the mass$m_1$

The centrifugal force exerted by mass $m_1$ on the shaft is given by,

$$\mathbf{F_{c1} = m_1\, \omega^2\, r_1} \text{ – – – – – – – – – – – – – – – – – – – – –} (1)$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force $F_{c1}$ , a balancing mass $m_2$ may be attached in the same plane of rotation of the disturbing mass $m_1$ such that the centrifugal forces due to the two masses are equal and opposite.

Let,

$r_2$ = radius of rotation of the mass $m_2$

= distancebetweenthe axis of rotationof the shaft and the centreof gravity of themassm$_2$

Therefore the centrifugal force due to mass $m_2$ will be,

$$F_{c2} = m_2 \omega^2 r_2 \text{ ------------------- } (2)$$

Equating equations (1) and (2), we get

$$F_{c1} = F_{c2}$$

$$m_1 \omega_2 r_1 = m_2 \omega_2 r_2 \quad \text{or } m_1 r_1 = m_2 r_2 \text{ ---------------- } (3)$$

The product $\mathbf{m}_2\ \mathbf{r}_2$ can be split up in any convenient way. As for as possible the radius of rotation of mass $m_2$ that is $r_2$ is generally made large in order to reduce the balancing mass $m_2$.

**CASE 2:**

**BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.**

There are two possibilities while attaching two balancing masses:

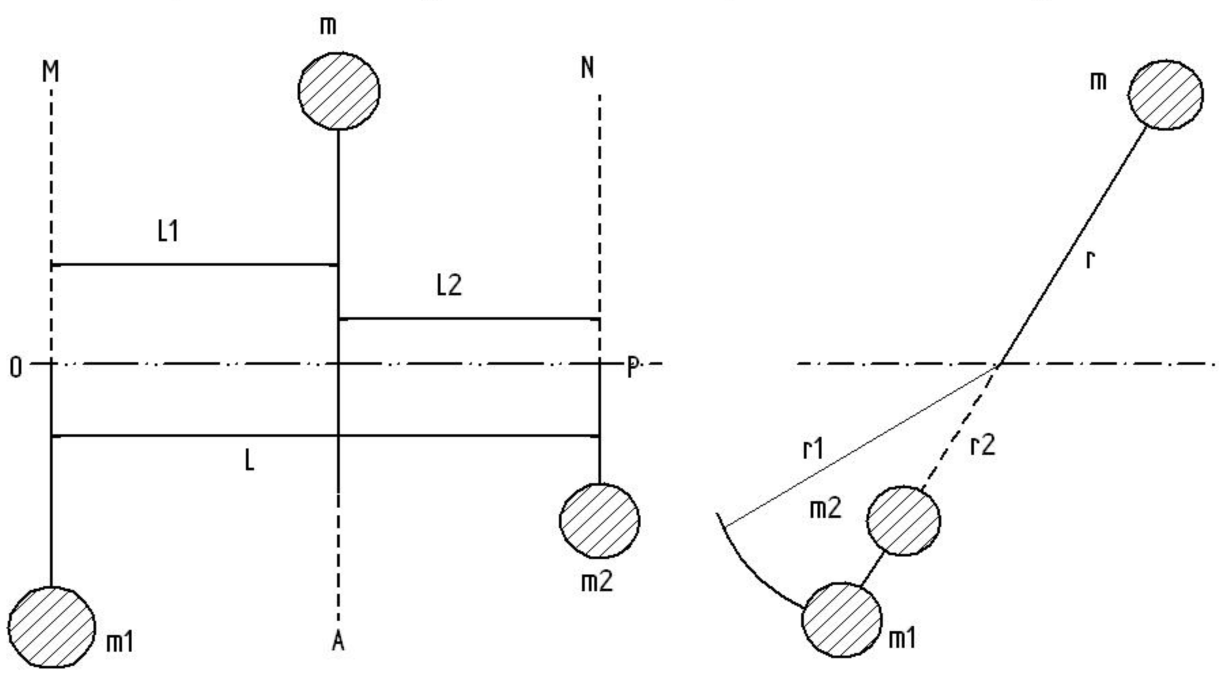
1. **The plane of the disturbing mass may be in between the planes of the two balancing masses.**
2. **The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.**

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

**CASE 2(I):**

**THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.**

The plane of the disturbing mass lies inbetween the planes of the two balancing masses




Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses $m_1$ and $m_2$ lying in two different planes M and N which are parallel to the plane A as shown.

Let r, $r_1$ and $r_2$ be the radii of rotation of the masses in planes A, M and N respectively.
Let $L_1$, $L_2$ and L be the distance between A and M, A and N, and M and N respectively. Now,
The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\,\omega^2\,r \text{ --------------------- } (1)$$

Similarly,
The centrifugal force exerted by the mass $m_1$ in plane M will be,

$$F_{c1} = m_1\,\omega^2\,r_1 \text{ -------------------- } (2)$$

And the centrifugal force exerted by the mass $m_2$ in plane N will be,

$$F_{c2} = m_2 \omega^2 r_2 \text{ -------------------- } (3)$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$

$$\text{or } m\omega^2 r = m_1 \omega^2 r_1 + m_2 \omega^2 r_2$$

$$\text{i.e. } mr = m_1 r_1 + m_2 r_2 \text{ ----------------- } (4)$$

Now, to determine the magnitude of balancing force in the plane 'M' or the dynamic force at the bearing 'O' of a shaft, take moments about ' P ' which is the point of intersection of the plane N and the axis of rotation.

Therefore,

$$F_{c1} \times L = F_c \times L_2$$

$$\text{or } m_1 \omega^2 r_1 \times L = m\omega^2 r \times L_2$$

Therefore,

$$m_1 r_1 L = mrL_2 \qquad \text{or } m_1 r_1 = mr\frac{L_2}{L} \text{ -------- } (5)$$

Similarly, in order to find the balancing force in plane 'N' or the dynamic force at the bearing 'P' of a shaft, take moments about ' O ' which is the point of intersection of the plane M and the axis of rotation.

Therefore,

$$F_{c2} \times L = F_c \times L_1$$

$$\text{or } m_2 \omega^2 r_2 \times L = m\omega^2 r \times L_1$$

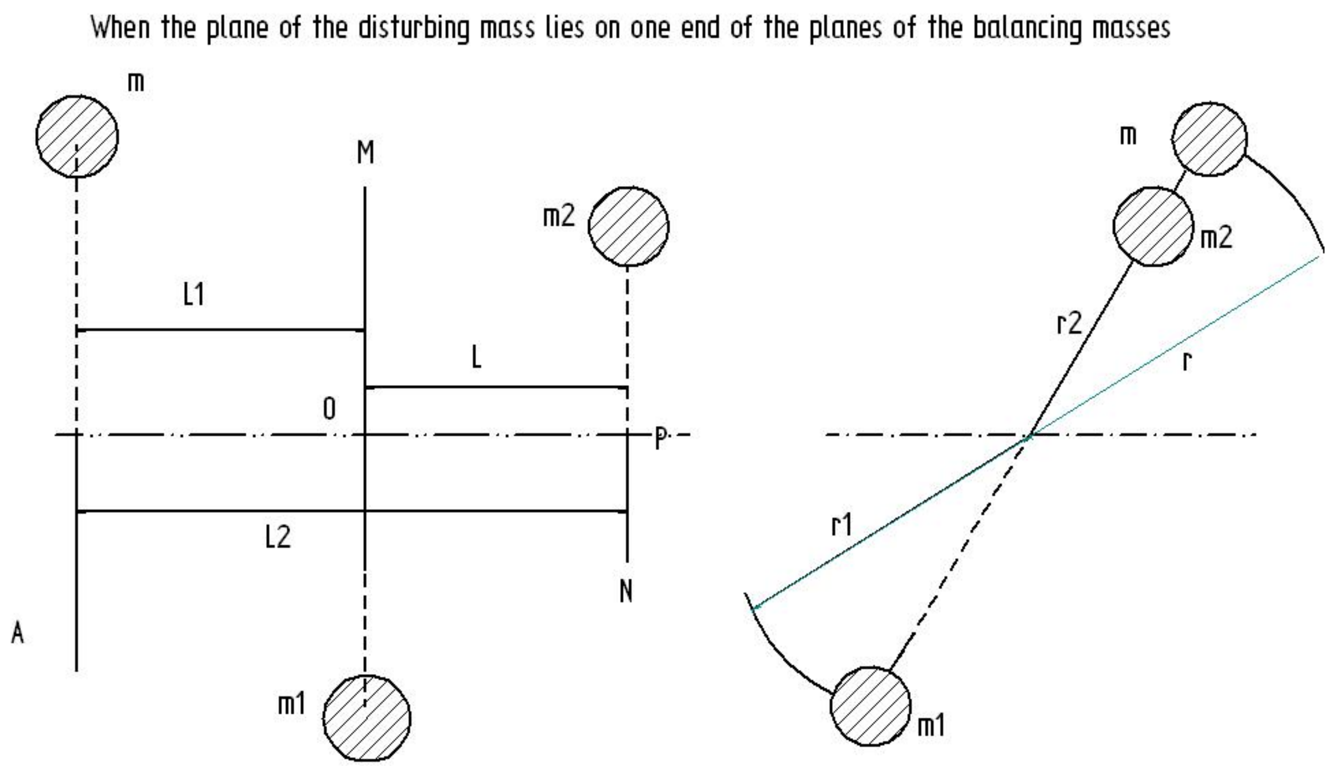
Therefore,

$$m_2 r_2 L = mrL_1 \qquad \text{or } m_2 r_2 = mr\frac{L_1}{L} \text{ -------- } (6)$$

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

**CASE 2(II):**

**WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.**




For static balancing,

$$F_{c1} = F_c + F_{c2}$$

$$\text{or } m_1\,\omega^2\,r_1 = m\omega^2\,r + m_2\,\omega^2\,r_2$$

$$\text{i.e. } m_1\,r_1 = mr + m_2\,r_2 \qquad (1)$$

For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.

To find the balancing force in the plane ‘M’ or the dynamic force at the bearing ‘O’ of a shaft, take moments about ‘P’. i.e.

$F_{c1} \times L = F_c \times L_2$

or $m_1 \omega^2 r_1 \times L = m\omega^2 r \times L_2$

Therefore,

$m_1 r_1 L = mrL_2$ or $m_1 r_1 = mr\frac{L_2}{L}$ --------(2)

Similarly, to find the balancing force in the plane 'N' , take moments about 'O', i.e.,

$F_{c2} \times L = F_c \times L_1$

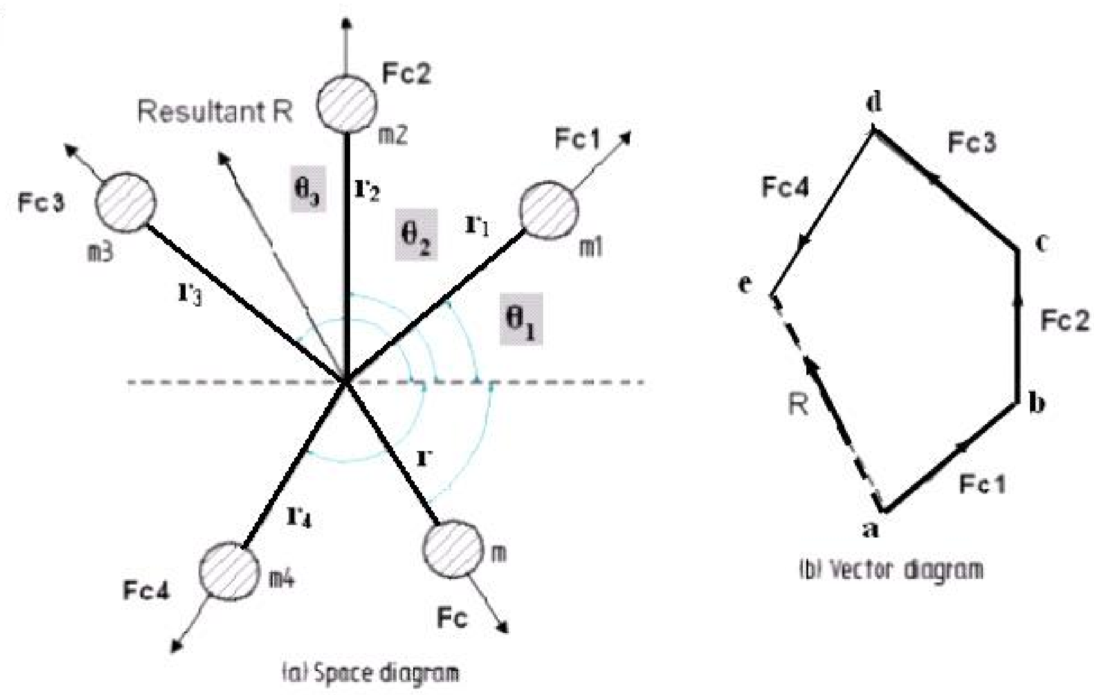
or $m_2 \omega^2 r_2 \times L = m\omega^2 r \times L_1$

Therefore,

$m_2 r_2 L = mrL_1$ or $m_2 r_2 = mr\frac{L_1}{L}$ --------(3)

**CASE 3:**

**BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE**




BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If $m_1$, $m_2$, $m_3$ and $m_4$ are the masses revolving at radii $r_1$, $r_2$, $r_3$ and $r_4$ respectively in the same plane.
The centrifugal forces exerted by each of the masses are $F_{c1}$, $F_{c2}$, $F_{c3}$ and $F_{c4}$ respectively.
Let F be the vector sum of these forces. i.e.

$$F = F_{c1} + F_{c2} + F_{c3} + F_{c4}$$
$$= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 \text{ ---------- } (1)$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight ( balance weight) of mass 'm' at radius 'r' to balance the rotor so that,

$$m_1 \omega_2 r_1 + m_2 \omega_2 r_2 + m_3 \omega_2 r_3 + m_4 \omega_2 r_4 + m \omega_2 r = 0 \text{ ---------- } (2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m r = 0 \text{ ----------------- } (3)$$

The magnitude of either 'm' or 'r' may be selected and the other can be calculated. In general, if $\sum \mathbf{m}_i \mathbf{r}_i$ is the vector sum of $\mathbf{m}_1 \mathbf{r}_1$ , $\mathbf{m}_2 \mathbf{r}_2$ , $\mathbf{m}_3 \mathbf{r}_3$ , $\mathbf{m}_4 \mathbf{r}_4$ etc, then,

$$\sum m_i r_i + mr = 0 \text{ ---------- } (4)$$

The above equation can be solved either analytically or graphically.

**1. Analytical Method:**

Procedure:

Step 1: Find out the centrifugal force or the product of mass and its radius of rotation exerted by each of masses on the rotating shaft, since $\omega^2$ is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.

Step 2: Resolve these forces into their horizontal and vertical components and find their sums. i.e.,

Sum of the horizontal components

$$= \sum_{i=1}^{n} m_i r_i \cos \theta_i = m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + \text{ ---------}$$

Sumof the vertical components

$$= \sum_{i=1}^{n} m_i r_i \sin \theta_i = m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + \text{ ---------}$$

Step 3: Determine the magnitude of the resultant centrifugal force

$$R=\sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2+\left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

Step 4: If $\theta$ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta=\frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i}$$

Step 5: The balancing force is then equal to the resultant force, but in opposite direction.
Step 6: Now find out the magnitude of the balancing mass, such that

$$R=mr$$

Where, m = balancing mass and r = its radius of rotation

**2. Graphical Method:**

Step 1:
Draw the space diagram with the positions of the several masses, as shown.

Step 2:
Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:
Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.
Let ab, bc, cd, de represents the forces $F_{c1}$, $F_{c2}$, $F_{c3}$ and $F_{c4}$ on the vector diagram. Draw 'ab' parallel to force $F_{c1}$ of the space diagram, at 'b' draw a line parallel to force $F_{c2}$. Similarly draw lines cd, de parallel to $F_{c3}$ and $F_{c4}$ respectively.

Step 4:
As per polygon law of forces, the closing side 'ae' represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:
The balancing force is then , equal and opposite to the resultant force.

Step 6:

Determine the magnitude of the balancing mass ( m ) at a given radius of rotation ( r ), such that,

$$F_c = m\omega^2 r$$

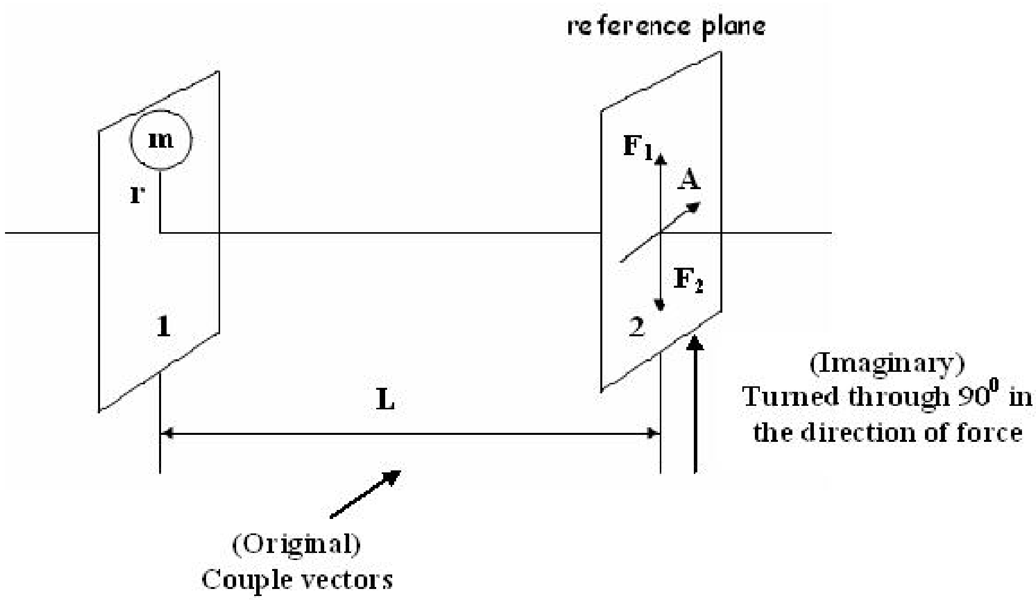
or

$$mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

**CASE 4:**

**BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES**

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.



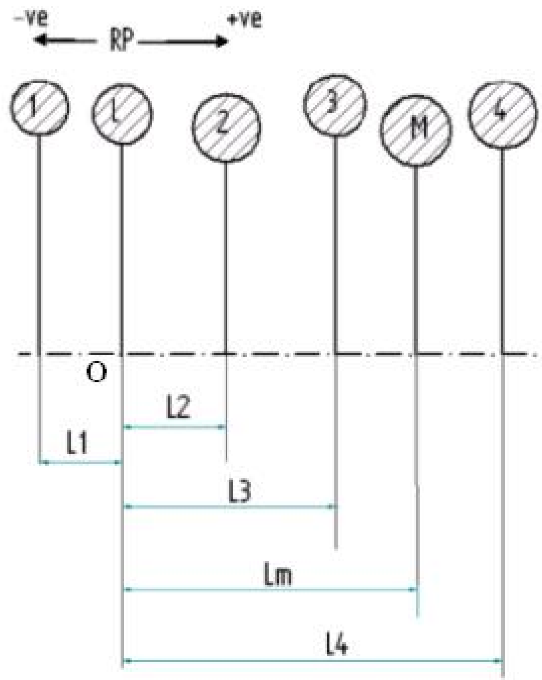

When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.

In order to have a complete balance of the several revolving masses in different planes,

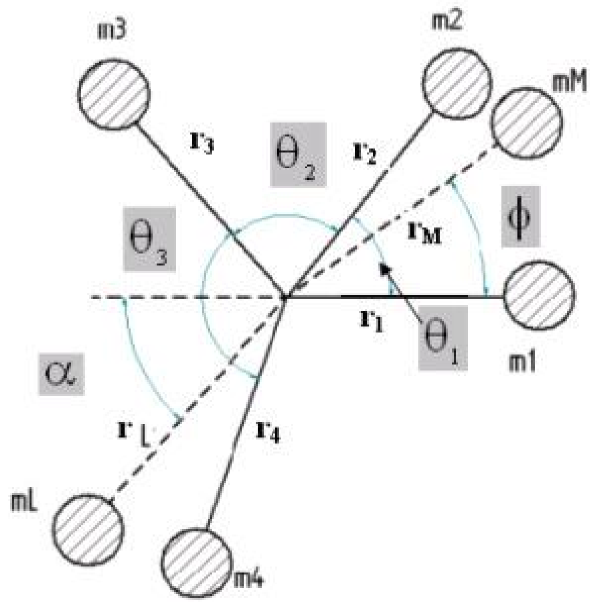
1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:
Consider four masses $m_1$, $m_2$, $m_3$ and $m_4$ attached to the rotor at radii $r_1$, $r_2$, $r_3$ and $r_4$ respectively. The masses $m_1$, $m_2$, $m_3$ and $m_4$ rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses



(b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at ‘O’ so that the distance of the planes 1, 2, 3 and 4 from ‘O’ are $L_1$, $L_2$ , $L_3$ and $L_4$ respectively. The reference plane chosen is plane ‘L’. Choose another plane ‘M’ between plane 3 and 4 as shown.

Plane ‘M’ is at a distance of $L_m$ from the reference plane ‘L’. The distances of all the other planes to the left of ‘L’ may be taken as negative( -ve) and to the right may be taken as positive (+ve).

The magnitude of the balancing masses $m_L$ and $m_M$ in planes L and M may be obtained by following the steps given below.

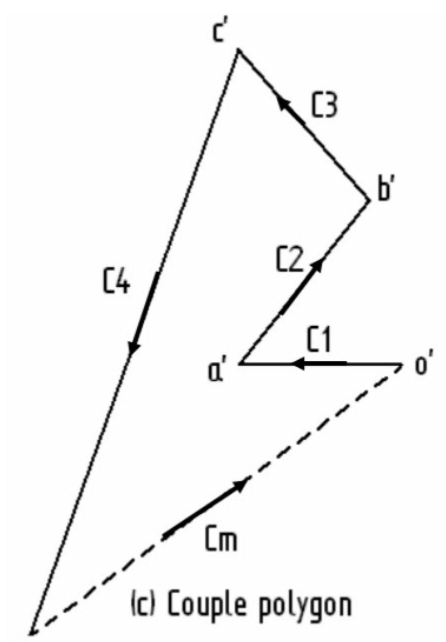
Step 1:
Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

| Plane 1 | Mass (m) 2 | Radius (r) 3 | Centrifugal force/ω$^2$ (m r) 4 | Distance from Ref. plane 'L' (L) 5 | Couple/ ω$^2$ (m r L) 6 |
|---|---|---|---|---|---|
| 1 | $m_1$ | $r_1$ | $m_1 r_1$ | $-L_1$ | $-m_1 r_1 L_1$ |
| L | $m_L$ | $r_L$ | $m_L r_L$ | 0 | 0 |
| 2 | $m_2$ | $r_2$ | $m_2 r_2$ | $L_2$ | $m_2 r_2 L_2$ |
| 3 | $m_3$ | $r_3$ | $m_3 r_3$ | $L_3$ | $m_3 r_3 L_3$ |
| M | $m_M$ | $r_M$ | $m_M r_M$ | $L_M$ | $m_M r_M L_M$ |
| 4 | $m_4$ | $r_4$ | $m_4 r_4$ | $L_4$ | $m_4 r_4 L_4$ |

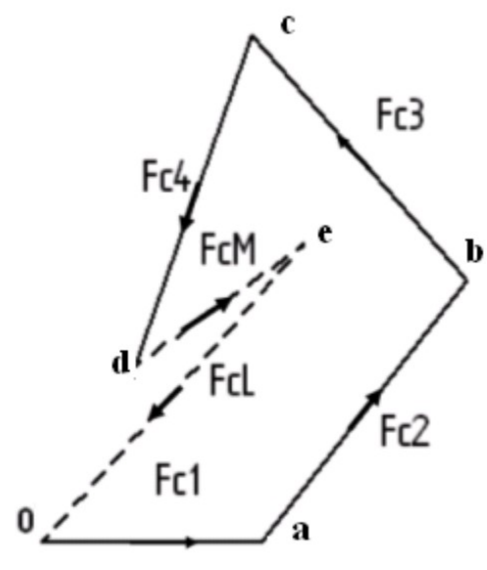
Step 2:

Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)

Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be '$m_M r_M L_M$'.



(c) Couple polygon



(d) Force polygon

The vector d 'o' on the couple polygon represents the balanced couple. Since the balanced couple $C_M$ is proportional to $m_M r_M L_M$ , therefore,

$$C_M = m_M r_M L_M = \text{vector } d'o'$$

$$\text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M L_M}$$

From this the value of $m_M$ in the plane M can be determined and the angle of inclination φ of this mass may be measured from figure (b).

Step 3:
Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with '$m_M r_M$'. The closing vector will be '$m_L r_L$'. This represents the balanced force. Since the balanced force is proportional to '$m_L r_L$',

$$m_L r_L = \text{vector eo}$$

$$\text{or} \quad m_L = \frac{\text{vector eo}}{r_L}$$

From this the balancing mass $m_L$ can be obtained in plane 'L' and the angle of inclination of this mass with the horizontal may be measured from figure (b).

## Problems and solutions

**Problem 1.**

Four masses A, B, C and D are attached to a shaft and revolve in the same plane. The masses are 12 kg, 10 kg, 18 kg and 15 kg respectively and their radii of rotations are 40 mm, 50 mm, 60 mm and 30 mm. The angular position of the masses B, C and D are $60^0$, $135^0$ and $270^0$ from mass A. Find the magnitude and position of the balancing mass at a radius of 100 mm.

Solution:
Given:

| Mass(m) kg | Radius(r) m | Centrifugal force/$\omega^2$ (m r) kg-m | Angle( θ ) |
|---|---|---|---|
| $m_A = 12$ kg (reference mass) | $r_A = 0.04$ m | $m_A r_A = 0.48$ kg-m | $\theta_A = 0^0$ |
| $m_B = 10$ kg | $r_B = 0.05$ m | $m_B r_B = 0.50$ kg-m | $\theta_B = 60^0$ |
| $m_C = 18$ kg | $r_C = 0.06$ m | $m_C r_C = 1.08$ kg-m | $\theta_C = 135^0$ |
| $m_D = 15$ kg | $r_D = 0.03$ m | $m_D r_D = 0.45$ kg-m | $\theta_D = 270^0$ |

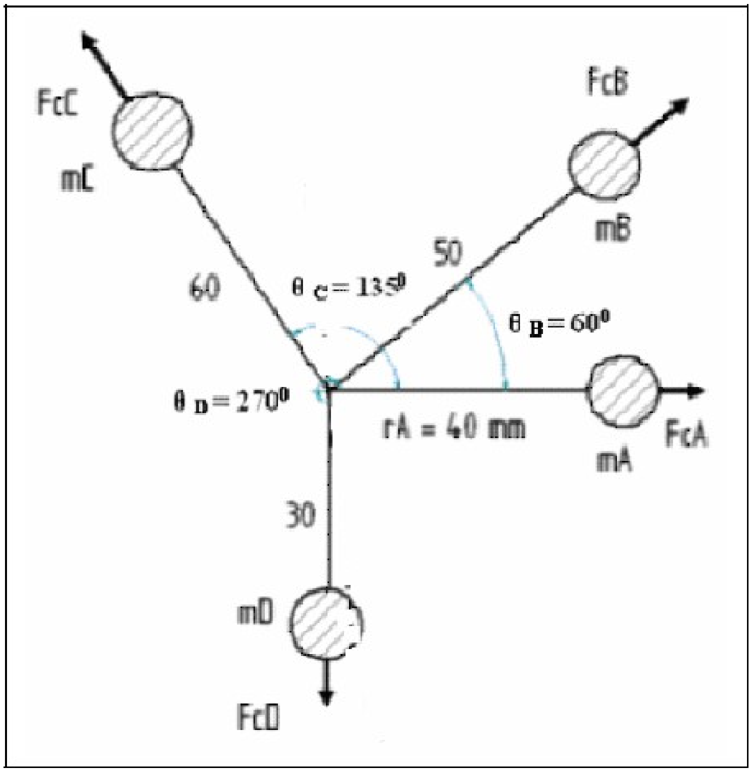
To determine the balancing mass 'm' at a radius of r = 0.1 m.

The problem can be solved by either analytical or graphical method.

## Analytical Method:

**Step 1:**

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$ .




Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

**Step 2:**

Resolve the centrifugal forces horizontally and vertically and find their sum.

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 0.48 \times \cos 0^0 + 0.50 \times \cos 60^0 + 1.08 \times \cos 135^0 + 0.45 \times \cos 270^0$$

$$= 0.48 + 0.25 + (-0.764) + 0 = -0.034 \text{ kg-m} \quad \text{----------} (1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 0.48 \times \sin 0^0 + 0.50 \times \sin 60^0 + 1.08 \times \sin 135^0 + 0.45 \times \sin 270^0$$

$$= 0 + 0.433 + 0.764 + (-0.45) = 0.747 kg-m \quad ---------- (2)$$

**Step 3:**

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(-0.034)^2 + (0.747)^2} = 0.748 kg-m$$

**Step 4:**

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = mr = 0.748 kg-m$$

$$\text{Therefore, } m = \frac{R}{r} = \frac{0.748}{0.1} = 7.48 kg \text{ Ans}$$

Where, m = balancing mass and r = its radius of rotation

**Step 5:**

Determine the position of the balancing mass 'm'.

If $\theta$ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{0.747}{-0.034} = -21.97$$

and $\theta = -87.4^0$ or $92.6^0$

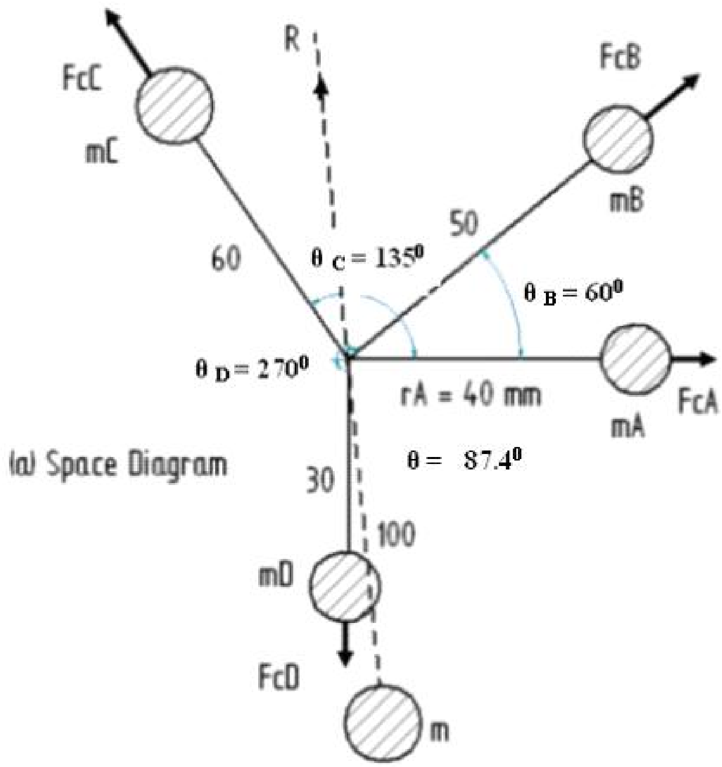
Remember ALL STUDENTS TAKE COPY i.e. in first quadrant all angles **( sin θ , cos θ and tan θ )** are positive, in second quadrant only **sin θ** is positive, in third quadrant only **tan θ** is positive and in fourth quadrant only **cos θ** is positive.

Since numerator is positive and denominator is negative, the resultant force makes with the horizontal, an angle (measured in the counter clockwise direction)

$$\theta = 92.6^0$$

The balancing force is then equal to the resultant force, but in opposite direction.

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is $\theta_M = 87.4^0$ angle measured in the clockwise direction.

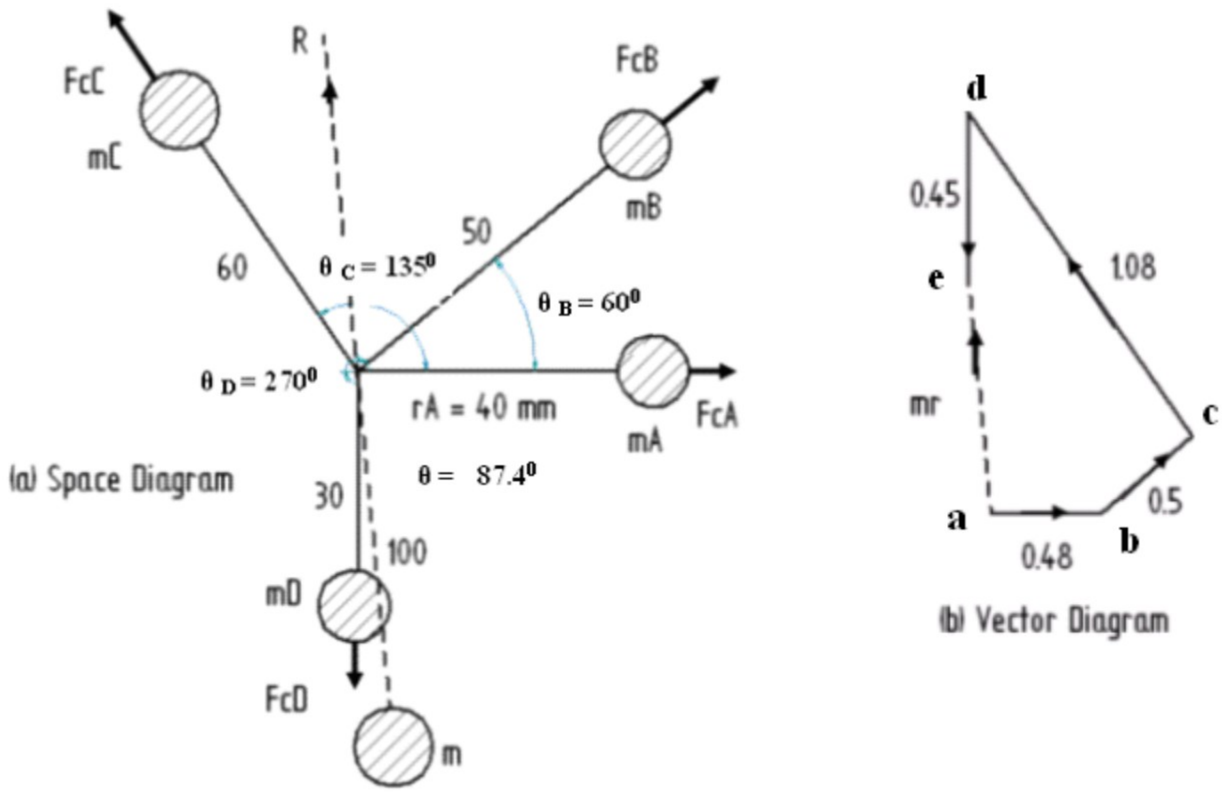



## Graphical Method:

### Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles( Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$ ).




**Step 2:**

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line 'ab' parallel to force $F_{CA}$ (or the product $m_A r_A$ to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to $F_{CB}$ (or the product $m_B r_B$). Similarly draw lines 'cd', 'de' parallel to $F_{CC}$ (or the product $m_C r_C$) and $F_{CD}$ (or the product $m_D r_D$) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

**Step 3:**

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore, } m = \frac{R}{r} = 7.48 \text{ kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.

**Problem 2:**

The four masses A, B, C and D are 100 kg, 150 kg, 120 kg and 130 kg attached to a shaft and revolve in the same plane. The corresponding radii of rotations are 22.5 cm, 17.5 cm,
25 cm and 30 cm and the angles measured from A are $45^0$, $120^0$ and $255^0$. Find the position and magnitude of the balancing mass, if the radius of rotation is 60 cm.

Solution:

## Analytical Method:

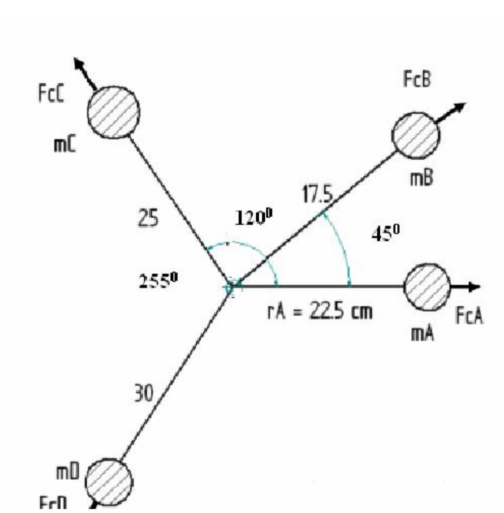
Given:

| Mass(m) kg | Radius(r) m | Centrifugal force/$\omega^2$ (m r) kg-m | Angle( $\theta$ ) |
|---|---|---|---|
| $m_A$ = 100 kg | $r_A$ = 0.225 m | $m_A r_A$ = 22.5 kg-m | $\theta_A = 0^0$ |
| $m_B$ = 150 kg | $r_B$ = 0.175 m | $m_B r_B$ = 26.25 kg-m | $\theta_B = 45^0$ |
| $m_C$ = 120 kg | $r_C$ = 0.250 m | $m_C r_C$ = 30 kg-m | $\theta_C = 120^0$ |
| $m_D$ = 130 kg | $r_D$ = 0.300 m | $m_D r_D$ = 39 kg-m | $\theta_D = 255^0$ |
| m =? | r = 0.60 | | $\theta$ = ? |

Step 1:

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.



**Step 2:**

Resolve the centrifugal forces horizontally and vertically and find their sum.

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 22.5 \times \cos 0^0 + 26.25 \times \cos 45^0 + 30 \times \cos 120^0 + 39 \times \cos 255^0$$

$$= 22.5 + 18.56 + (-15) + (-10.1) = 15.97 \text{ kg} - \text{m} \quad --------- (1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 22.5 \times \sin 0^0 + 26.25 \times \sin 45^0 + 30 \times \sin 120^0 + 39 \times \sin 255^0$$

$$= 0 + 18.56 + 25.98 + (-37.67) = 6.87 \text{ kg} - \text{m} \quad --------- (2)$$

**Step 3:**

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(15.97)^2 + (6.87)^2} = 17.39 \text{ kg} - \text{m}$$

**Step 4:**

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = m\,r = 17.39 \text{ kg} - \text{m}$$

$$\text{Therefore, } m = \frac{R}{r} = \frac{17.39}{0.60} = 28.98 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

**Step 5:**

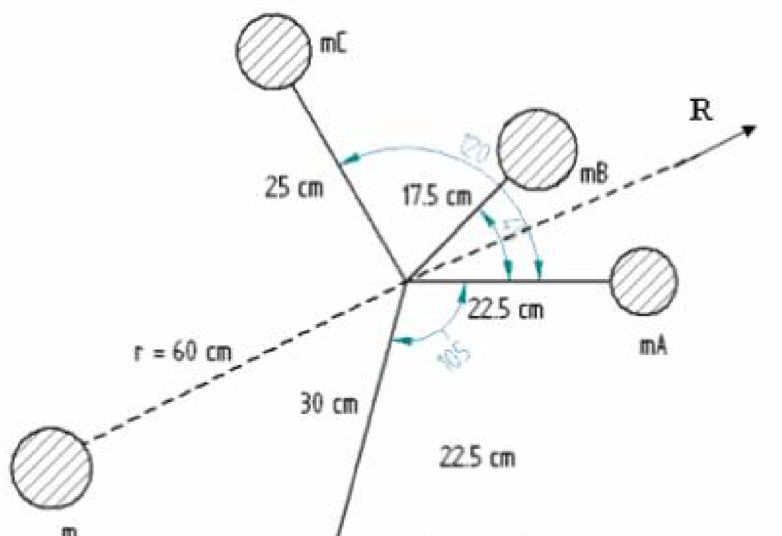
Determine the position of the balancing mass 'm'.

If $\theta$ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i\, r_i \sin\theta_i}{\sum_{i=1}^{n} m_i\, r_i \cos\theta_i} = \frac{6.87}{15.97} = 0.4302$$

and $\theta = 23.28^0$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203.28^0$ angle measured in the counter clockwise direction.




(a) Space diagram

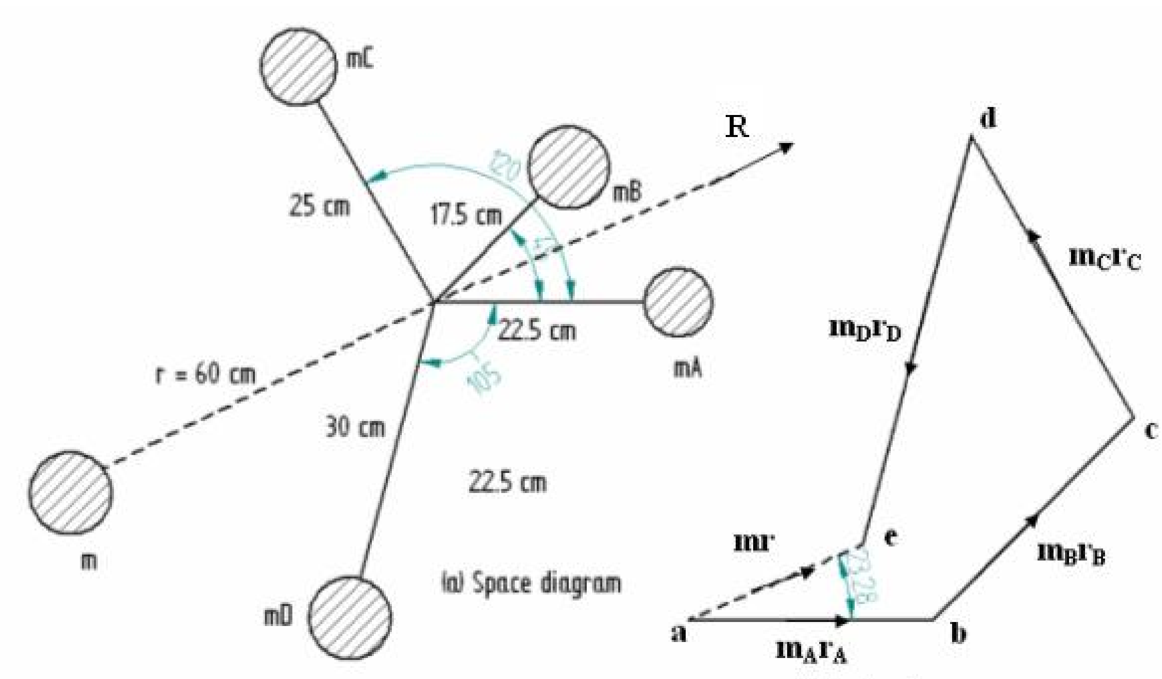
## Graphical Method:

### Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

**Step 2:**

Draw the space diagram or angular position of the masses taking the actual angles (Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$ ).




(a) Space diagram

(b) Vector diagram

Step 3:

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line 'ab' parallel to force $F_{CA}$ (or the product $m_A r_A$ to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to $F_{CB}$ (or the product $m_B r_B$). Similarly draw lines 'cd', 'de' parallel to $F_{CC}$ (or the product $m_C r_C$) and $F_{CD}$ (or the product $m_D r_D$) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

**Step 4:**

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore, } m = \frac{R}{r} = 29 \text{kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203^0$ angle measured in the counter clockwise direction.

**Problem 3:**
A rotor has the following properties.

| Mass | magnitude | Radius | Angle | Axial distance from first mass |
|---|---|---|---|---|
| 1 | 9 kg | 100 mm | $\theta_A = 0^0$ | - |
| 2 | 7 kg | 120 mm | $\theta_B = 60^0$ | 160 mm |
| 3 | 8 kg | 140 mm | $\theta_C = 135^0$ | 320 mm |
| 4 | 6 kg | 120 mm | $\theta_D = 270^0$ | 560 mm |

If the shaft is balanced by two counter masses located at 100 mm radii and revolving in planes midway of planes 1 and 2, and midway of 3 and 4, determine the magnitude of the masses and their respective angular positions.

Solution:

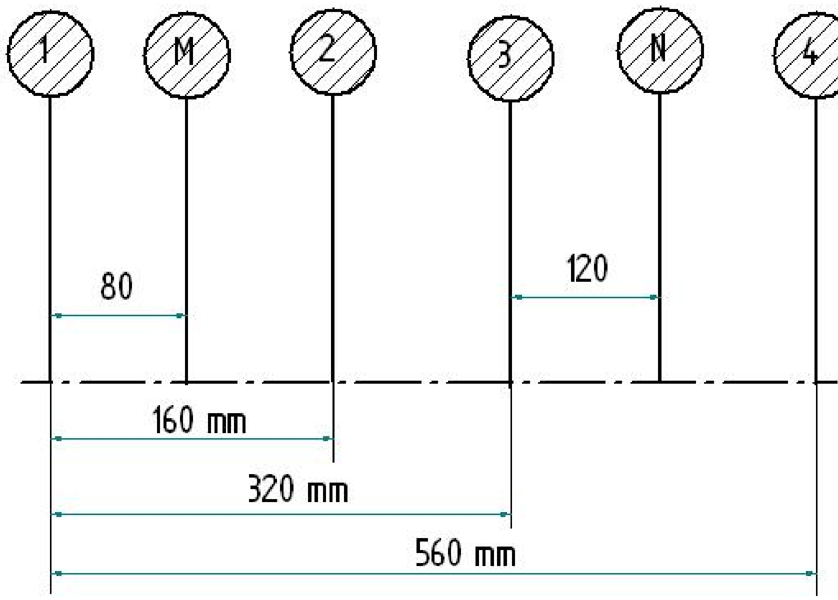
**Analytical Method:**

| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/$\omega^2$ (m r) kg-m 4 | Distance from Ref. plane 'M' m 5 | Couple/$\omega^2$ (m r L) kg-$m^2$ 6 | Angle $\theta$ 7 |
|---|---|---|---|---|---|---|
| 1 | 9.0 | 0.10 | $m_1 r_1 = 0.9$ | -0.08 | -0.072 | $0^0$ |
| M | $m_M$ = ? | 0.10 | $m_M r_M = 0.1 m_M$ | 0 | 0 | **$\theta_M$ =?** |
| 2 | 7.0 | 0.12 | $m_2 r_2 = 0.84$ | 0.08 | 0.0672 | $60^0$ |
| 3 | 8.0 | 0.14 | $m_3 r_3 = 1.12$ | 0.24 | 0.2688 | $135^0$ |
| N | $m_N$ = ? | 0.10 | $m_N r_N = 0.1 m_N$ | 0.36 | $m_N r_N l_N = 0.036 m_N$ | **$\theta_N$ =?** |
| 4 | 6.0 | 0.12 | $m_4 r_4 = 0.72$ | 0.48 | 0.3456 | $270^0$ |

For dynamic balancing the conditions required are,

$\sum mr + m_M r_M + m_N r_N = 0$ ----------(I) for force balance

$\sum mrl + m_N r_N l_N = 0$ --------------(II) for couple balance




(a) Position of planes of masses

**Step 1:**
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$\sum mrl\cos\theta + m_N r_N l_N \cos\theta_N = 0$ Onsubstitution we get

$-0.072\cos 0^0 + 0.0672\cos 60^0 + 0.2688\cos 135^0$

$+0.3456\cos 270^0 + 0.036\, m_N \cos\theta_N = 0$ i.e.

$0.036\, m_N \cos\theta_N = 0.2285 - - - - - (1)$

Sum of the vertical components gives,

$\sum mrl\sin\theta + m_N r_N l_N \sin\theta_N = 0$ On substitution we get

$-0.072\sin 0^0 + 0.0672\sin 60^0 + 0.2688\sin 135^0$

$+0.3456\sin 270^0 + 0.036\, m_N \sin\theta_N = 0$ i.e. $0.036\, m_N \sin\theta_N = 0.09733 - - - - - (2)$

Squaring and adding (1) and (2), we get

$$m_N r_N l_N = \sqrt{(0.2285)^2 + (0.09733)^2}$$

i.e., $0.036 m_N = 0.2484$

Therefore, $m_N = \frac{0.2484}{0.036} = 6.9\text{kg}$ Ans

Dividing (2) by (1), we get

$$\tan\theta_N = \frac{0.09733}{0.2285} \quad \text{and } \theta_N = 23.07^0$$

**Step 2:**

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_M r_M \cos\theta_M + m_N r_N \cos\theta_N = 0$$

On substitution we get

$$0.9\cos 0^0 + 0.84\cos 60^0 + 1.12\cos 135^0 + 0.72\cos 270^0 +$$

$$m_M r_M \cos\theta_M + 0.1 \times 6.9 \times \cos 23.07^0 = 0$$

i.e. $m_M r_M \cos\theta_M = -1.1629 - - - - - (3)$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_M r_M \sin\theta_M + m_N r_N \sin\theta_N = 0$$

On substitution we get

$$0.9\sin 0^0 + 0.84\sin 60^0 + 1.12\sin 135^0 + 0.72\sin 270^0 +$$

$$m_M r_M \sin\theta_M + 0.1 \times 6.9 \times \sin 23.07^0 = 0$$

i.e. $m_M r_M \sin\theta_M = -1.0698 - - - - - (4)$

Squaring and adding (3) and (4), we get

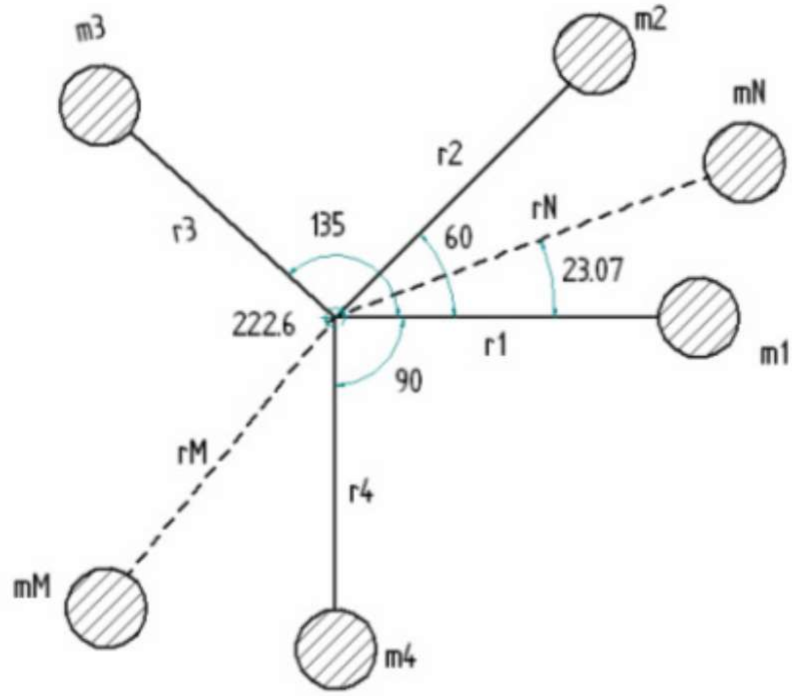
$$m_M r_M = \sqrt{(-1.1629)^2 + (-1.0698)^2}$$

i.e., $0.1 m_M = 1.580$

Therefore, $m_M = \frac{1.580}{0.1} = 15.8$ kg Ans

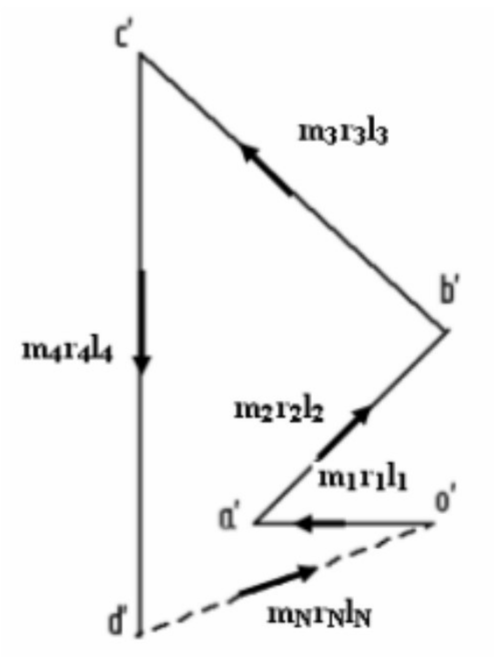
Dividing (4) by (3), we get

$$\tan\theta_M = \frac{-1.0698}{-1.1629} \quad \text{and } \theta_M = 222.61^0 \quad \text{Ans}$$

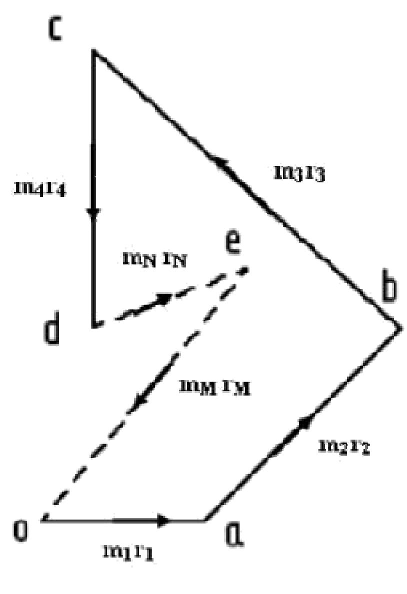


(b) Angular position of masses

## Graphical Solution:



(c) Couple polygon



(d) Force polygon

Problem 4:
The system has the following data.

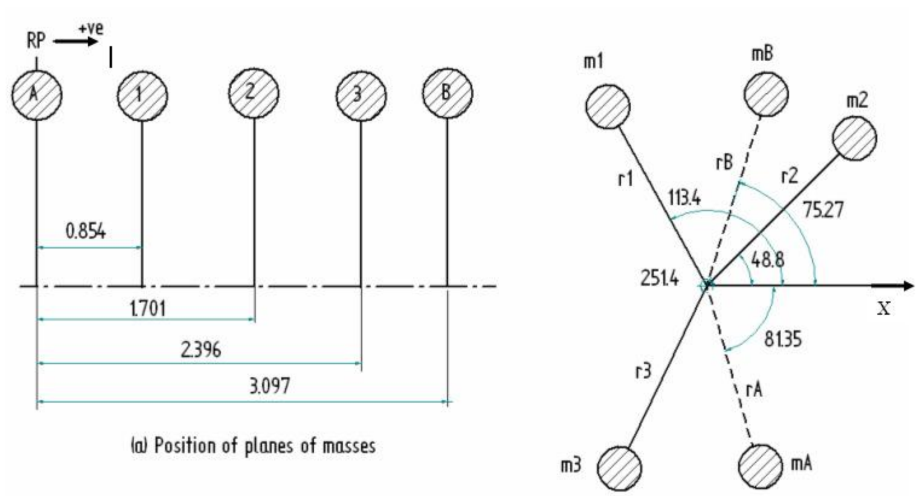
| | |
|---|---|
| $m_1$ =1.2 kg | $r_1$ = 1.135 m @ ∠ 113.4$^0$ |
| $m_1$ =1.8 kg | $r_2$ = 0.822 m @ ∠ 48.8$^0$ |
| $m_1$ = 2.4 kg | $r_3$ = 1.04 m @ ∠ 251.4$^0$ |

The distances of planes in metres from plane A are:

$l_1 = 0.854,\ l_2 = 1.701, l_3 = 2.396, l_B = 3.097$

Find the mass-radius products and their angular locations needed to dynamically balance the system using the correction planes A and B.

**Solution:** **Analytical Method**




(a) Position of planes of masses

(b) Angular position of masses

| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/ω$^2$ (m r) kg-m 4 | Distance from Ref. plane 'A' m 5 | Couple/ ω$^2$ (m r L) kg-m$^2$ 6 | Angle θ 7 |
|---|---|---|---|---|---|---|
| A | $m_A$ | $r_A$ | $m_A\ r_A$ =? | 0 | 0 | $\theta_A$ =? |
| 1 | 1.2 | 1.135 | 1.362 | 0.854 | 1.163148 | 113.4$^0$ |
| 2 | 1.8 | 0.822 | 1.4796 | 1.701 | 2.5168 | 48.8$^0$ |
| 3 | 2.4 | 1.04 | 2.496 | 2.396 | 5.9804 | 251.4$^0$ |
| B | $m_B$ | $r_B$ | $m_B\ r_B$ =? | 3.097 | 3.097 $m_B\ r_B$ | $\theta_B$ =? |

**Step 1:**

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_B r_B l_B \cos\theta_B = 0$$

Onsubstitution we get

$$1.163148\cos 113.4^0 + 2.5168\cos 48.8^0 + 5.9804\cos 251.4^0 + 3.097\, m_B r_B \cos\theta_B = 0$$

$$\text{i.e. } m_B r_B \cos\theta_B = \frac{0.71166}{3.097} \text{ -----(1)}$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_B r_B l_B \sin\theta_B = 0$$

Onsubstitution we get

$$1.163148\sin 113.4^0 + 2.5168\sin 48.8^0 + 5.9804\sin 251.4^0 + 3.097\, m_B r_B \sin\theta_B = 0$$

$$\text{i.e. } m_B r_B \sin\theta_B = \frac{2.7069}{3.097} \text{ ------ (2)}$$

Squaring and adding (1) and (2), we get

$$m_B r_B = \sqrt{\left(\frac{0.71166}{3.097}\right)^2 + \left(\frac{2.7069}{3.097}\right)^2}$$

$$= 0.9037 \text{kg} - \text{m}$$

Dividing (2) by (1), we get

$$\tan\theta_B = \frac{2.7069}{0.71166} \text{ and } \theta_B = 75.27^0 \quad \text{Ans}$$

**Step 2:**

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_A r_A \cos\theta_A + m_B r_B \cos\theta_B = 0$$

On substitution we get

$$1.362\cos 113.4^0 + 1.4796\cos 48.8^0 + 2.496\cos 251.4^0$$

$$+ m_A r_A \cos\theta_A + 0.9037\cos 75.27^0 = 0$$

Therefore

$$m_A r_A \cos\theta_A = 0.13266 - - - - - - - - - - (3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_A r_A \sin\theta_A + m_B r_B \sin\theta_B = 0$$

On substitution we get

$$1.362\sin 113.4^0 + 1.4796\sin 48.8^0 + 2.496\sin 251.4^0$$

$$+ m_A r_A \sin\theta_A + 0.9037\sin 75.27^0 = 0$$

Therefore

$$m_A r_A \sin\theta_A = -0.87162 - - - - - - - - - - (4)$$

Squaring and adding (3) and (4), we get

$$m_A r_A = \sqrt{(0.13266)^2 + (-0.87162)^2}$$
$$= 0.8817 \text{ kg} - \text{m}$$

Dividing (4) by (3), we get

$$\tan\theta_A = \frac{-0.87162}{0.13266} \quad \text{and } \theta_A = -81.35^0 \text{ Ans}$$

Problem 5:

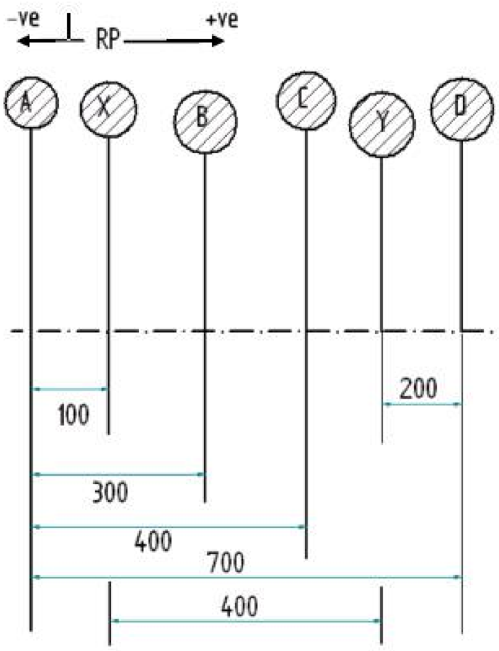
A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B $45^0$, B to C $70^0$ and C to D $120^0$. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

## Graphical solution:

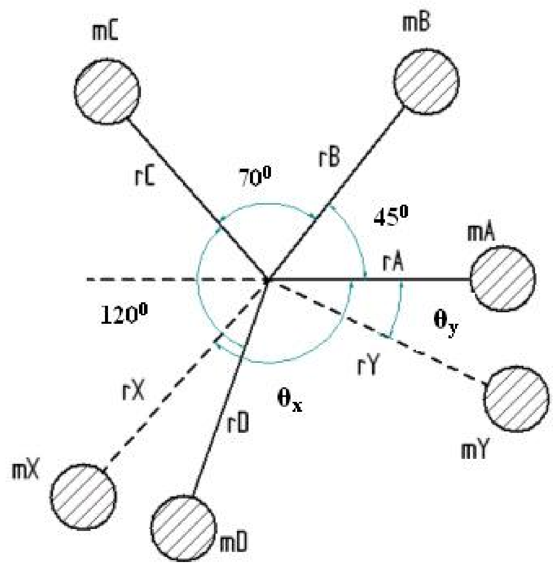
Let, $m_X$ be the balancing mass placed in plane X and $m_Y$ be the balancing mass placed in plane Y which are to be determined.

**Step 1:**

Draw the position of the planes as shown in figure (a).




(a) position of planes of masses




(b) Angular position of masses

Let X be the reference plane (R.P.). The distances of the planes to the right of the plane X are taken as positive (+ve) and the distances of planes to the left of X plane are taken as negative(-ve). The data may be tabulated as shown

Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'm r' can be calculated and tabulated. Similarly the magnitude of the couples are proportional to the product of the mass , its radius and the axial distance from the reference plane, the product 'm r l' can be calculated and tabulated as shown.

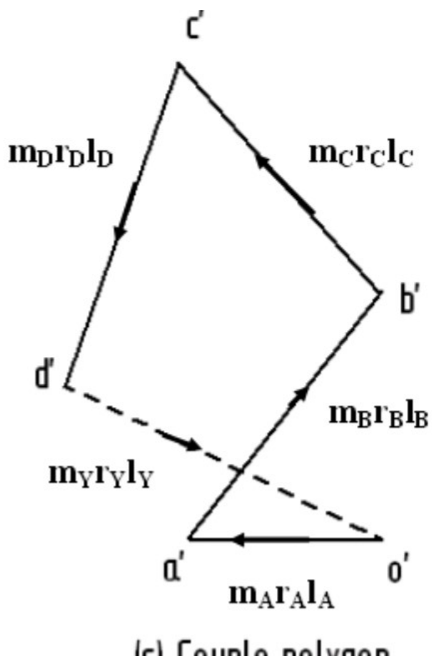
| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/$\omega^2$ (m r) kg-m 4 | Distance from Ref. plane 'X' m 5 | Couple/$\omega^2$ (m r L) kg-$m^2$ 6 | Angle $\theta$ 7 |
|---|---|---|---|---|---|---|
| A | 200 | 0.08 | $m_A r_A$ = 16 | -0.10 | -1.60 | - |
| X | $m_X$ =? | 0.10 | $m_X r_X$ = 0.1 $m_X$ | 0 | 0 | $\theta_X$ =? |
| B | 300 | 0.07 | $m_B r_B$ = 21 | 0.20 | 4.20 | A to B $45^0$ |
| C | 400 | 0.06 | $m_C r_C$ = 24 | 0.30 | 7.20 | B to C $70^0$ |
| Y | $m_Y$ =? | 0.10 | $m_Y r_Y$ = 0.1 $m_Y$ | 0.40 | $m_Y r_Y l_Y = 0.04\, m_Y$ | $\theta_Y$ =? |
| D | 200 | 0.08 | $m_D r_D$ = 16 | 0.60 | 9.60 | C to D $120^0$ |

**Step 2:**

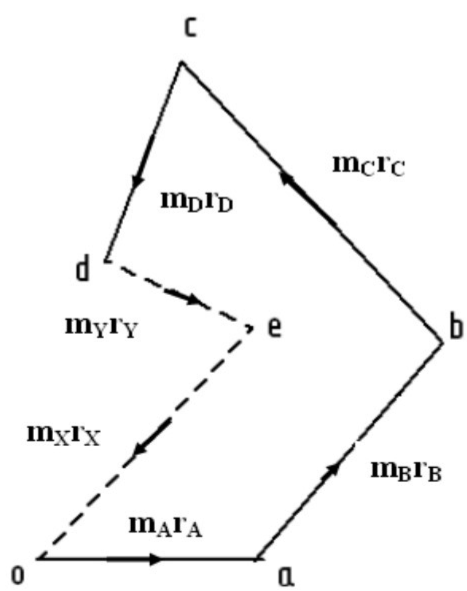
Assuming the mass A as horizontal draw the sketch of angular position of masses as shown in figure (b).

**Step 3:**

Draw the couple polygon to some suitable scale by taking the values of 'm r l' (column no. 6) of the table as shown in figure (c).



(c) Couple polygon



(d) Force polygon

Draw line o'a' parallel to the radial line of mass $m_A$.
At a' draw line a'b' parallel to radial line of mass $m_B$.
Similarly, draw lines b'c', c'd' parallel to radial lines of masses $m_C$ and $m_D$ respectively.
Now, join d' to o' which gives the balanced couple.

We get, $0.04\ m_Y = \text{vector } d'o' = 7.3\text{kg} - m^2$
or $m_Y = 182.5$ kg Ans

**Step 4:**
To find the angular position of the mass $m_Y$ draw a line $om_Y$ in figure (b) parallel to d'o' of the couple polygon.

By measurement we get $\theta_Y = 12^0$ in the clockwise direction from $m_A$.

**Step 5:**
Now draw the force polygon by considering the values of 'm r' (column no. 4) of the table as shown in figure (d).
Follow the similar procedure of step 3. The closing side of the force polygon i.e. 'e o' represents the balanced force.

$$m_X\, r_X = \text{vector } eo = 35.5\text{kg} - m$$
$$\text{or } m_X = 355\text{kg Ans}$$

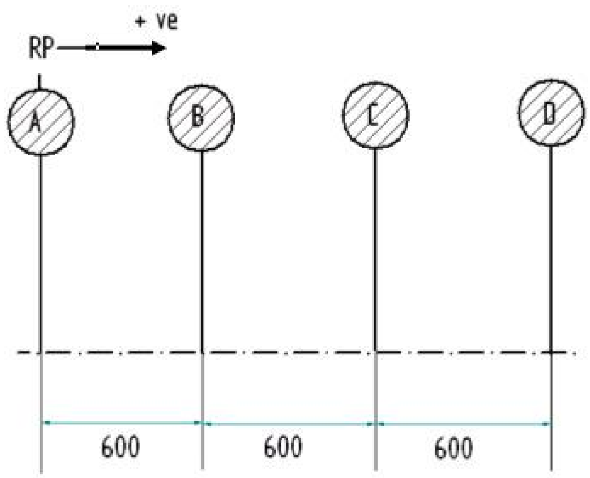
**Step 6:**
The angular position of $m_X$ is determined by drawing a line $om_X$ parallel to the line 'e o' of the force polygon in figure ( b). From figure (b) we get,
$\theta_X = 145^0$ , measured clockwise from $m_A$. Ans

**Problem 6:**
A, B, C and D are four masses carried by a rotating shaft at radii 100 mm, 125 mm, 200 mm and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg and 4 kg respectively. Find the required mass A and relative angular settings of the four masses so that the shaft shall be in complete balance.
Solution:

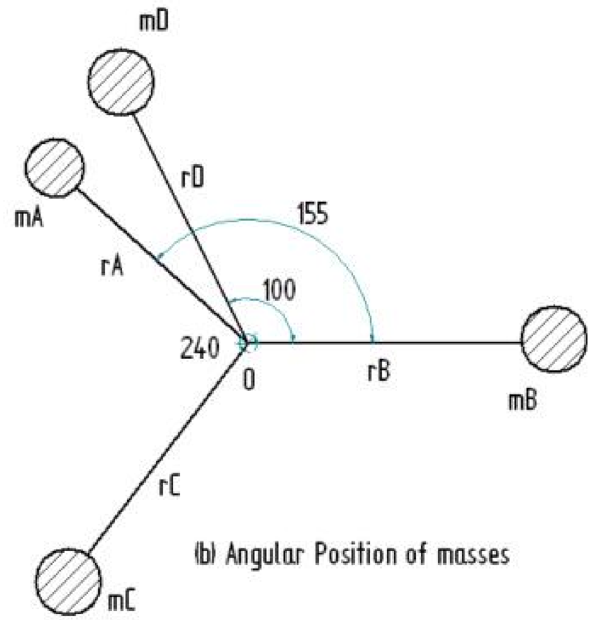
**Graphical Method:**

**Step 1:**
Let, $m_A$ be the balancing mass placed in plane A which is to be determined along with the relative angular settings of the four masses.
Let A be the reference plane (R.P.).
Assume the mass B as horizontal
Draw the sketch of angular position of mass $m_B$ (line $om_B$ ) as shown in figure (b). The data may be tabulated as shown.

| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/$\omega^2$ (m r) kg-m 4 | Distance from Ref. plane 'A' m 5 | Couple/ $\omega^2$ (m r L) kg-$m^2$ 6 | Angle $\theta$ 7 |
|---|---|---|---|---|---|---|
| A (R.P.) | $m_A$= ? | 0.1 | $m_A r_A$ = 0.1 $m_A$ | 0 | 0 | $\theta_A$ =? |
| B | 10 | 0.125 | $m_B r_B$ = 1.25 | 0.6 | 0.75 | $\theta_B$ =0 |
| C | 5 | 0.2 | $m_C r_C$ = 1.0 | 1.2 | 1.2 | $\theta_C$ =? |
| D | 4 | 0.15 | $m_D r_D$ = 0.6 | 1.8 | 1.08 | $\theta_D$ =? |



(a) Position of planes of masses



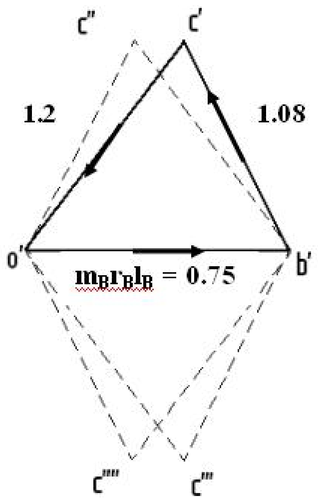
(b) Angular Position of masses

Step 2:
To determine the angular settings of mass C and D the couple polygon is to be drawn first as shown in fig (c). Take a convenient scale

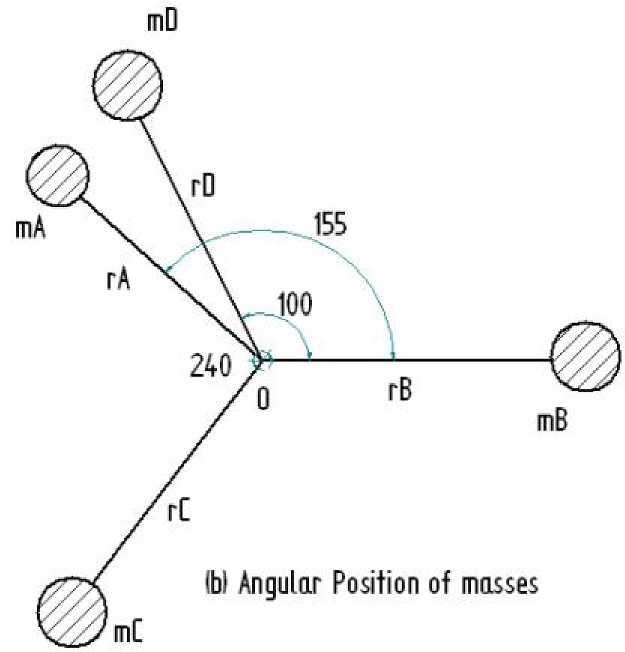
Draw a line o'b' equal to 0.75 kg-$m^2$ parallel to the line o$m_B$. At point o' and b' draw vectors o'c' and b'c' equal to 1.2 kg-$m^2$ and 1.08 kg-$m^2$ respectively. These vectors intersect at point c'.

**For the construction of force polygon there are four options.**

**Any one option can be used and relative to that the angular settings of mass C and D are determined.**




(c) Couple polygon




(b) Angular Position of masses

Step 3:

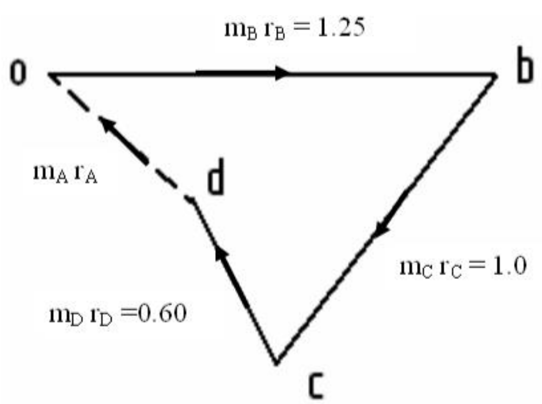
Now in figure (b), draw lines $om_C$ and $om_D$ parallel to o'c' and b'c' respectively.

From measurement we get,

$$\theta_D = 100^0 \quad \text{and} \quad \theta_C = 240^0 \ \text{Ans}$$

**Step 4:**

In order to find $m_A$ and its angular setting draw the force polygon as shown in figure (d).




(d) Force polygon

Closing side of the force polygon od represents the product $m_A$ $r_A$ . i.e.

$m_A \, r_A = 0.70$kg-m

Therefore, $m_A = \dfrac{0.70}{r_A} = 7$kg Ans

**Step 5:**

Now draw line $om_A$ parallel to od of the force polygon. By measurement, we get,

$$\theta_A = 155^0 \quad \text{Ans}$$

**Problem 7:**

A shaft carries three masses A, B and C. Planes B and C are 60 cm and 120 cm from A. A , B and C are 50 kg, 40 kg and 60 kg respectively at a radius of 2.5 cm. The angular position of mass B and mass C with A are $90^0$ and $210^0$ respectively. Find the unbalanced force and couple. Also find the position and magnitude of balancing mass required at 10 cm radius in planes L and M midway between A and B, and B and C.

**Solution:**

**Case (i):**

| Plane<br>1 | Mass<br>(m) kg<br>2 | Radius (r)<br>m<br>3 | Centrifugal force/$\omega^2$<br>(m r)<br>kg-m<br>4 | Distance from Ref. plane 'A'<br>m<br>5 | Couple/ $\omega^2$<br>(m r L)<br>kg-$m^2$<br>6 | Angle<br>$\theta$<br>7 |
|---|---|---|---|---|---|---|
| A<br>(R.P.) | 50 | 0.025 | $m_A r_A = 1.25$ | 0 | 0 | $\theta_A = 0^0$ |
| B | 40 | 0.025 | $m_B r_B = 1.00$ | 0.6 | 0.6 | $\theta_B = 90^0$ |
| C | 60 | 0.025 | $m_C r_C = 1.50$ | 1.2 | 1.8 | $\theta_C = 210^0$ |

**Analytical Method**

**Step 1:**
**Determination of unbalanced couple**
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta = 0.6\cos 90^0 + 1.8\cos 210^0 = -1.559 - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta = 0.6\sin 90^0 + 1.8\sin 210^0 = -0.3 - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$C_{unbalanced} = \sqrt{(-1.559)^2 + (-0.3)^2}$$
$$= 1.588 \text{ kg}-\text{m}^2$$

**Step 2:**
**Determination of unbalanced force**

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta = 1.25\cos 0^0 + 1.0\cos 90^0 + 1.5\cos 210^0$$
$$= 1.25 + 0 + (-1.299) = -0.049 - - - - - - - - - - (3)$$

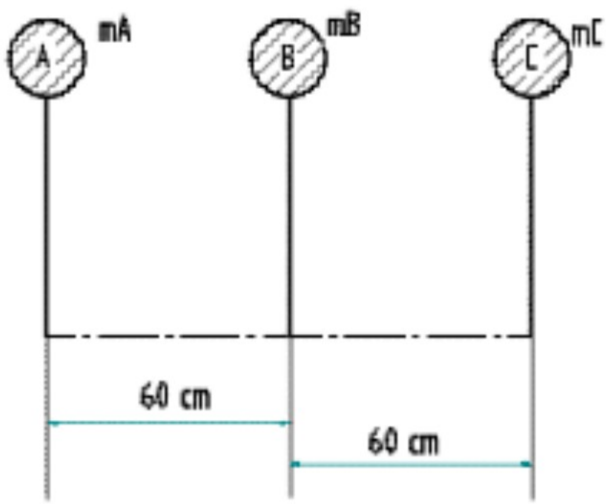
Sum of the vertical components gives,

$$\sum mr\sin\theta = 1.25\sin 0^0 + 1.0\sin 90^0 + 1.5\sin 210^0$$
$$= 0 + 1.0 + (-0.75) = 0.25 - - - - - - - - - - (4)$$

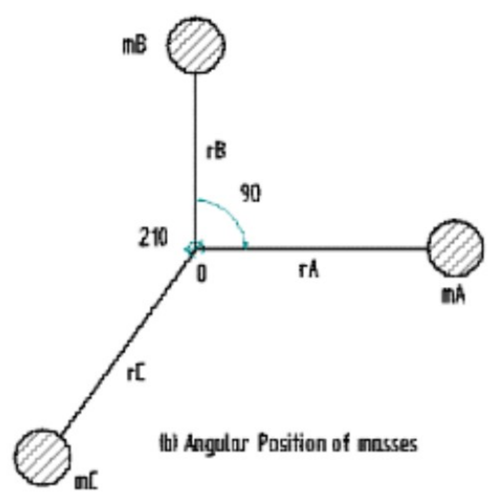
Squaring and adding (3) and (4), we get

$$F_{unbalanced} = \sqrt{(-0.049)^2 + (0.25)^2}$$
$$= 0.2548 \text{ kg}-\text{m}$$

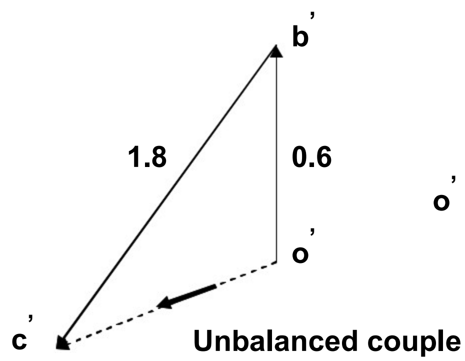
**Graphical solution:**



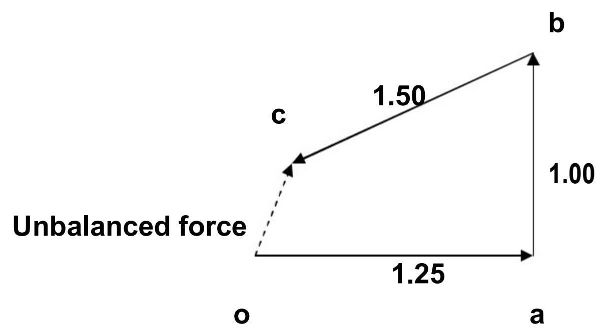

(a) Position of planes of masses




(b) Angular Position of masses

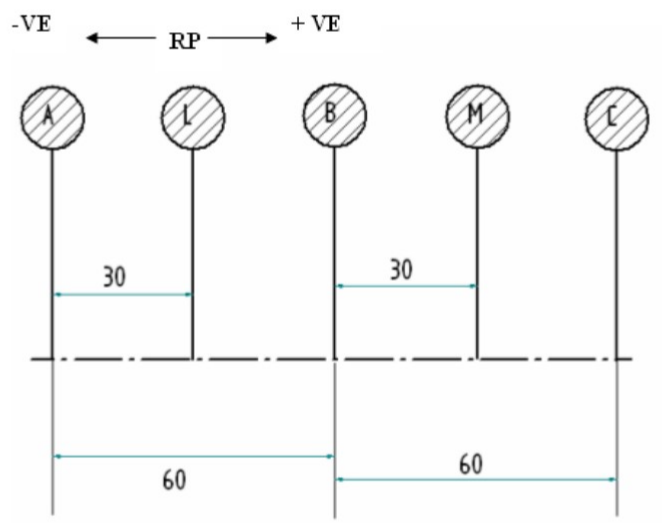



Couple polygon

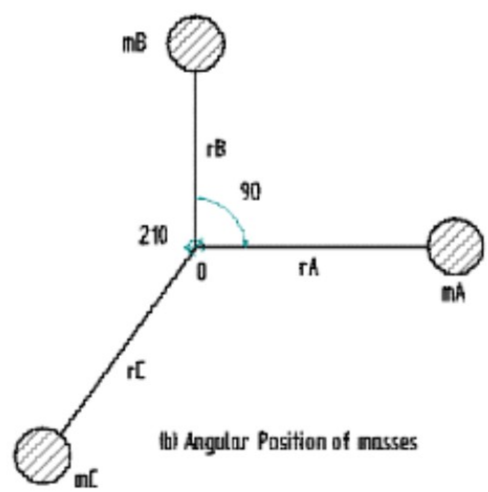



Force polygon

**Case (ii):**




(a) Position of planes of masses




(b) Angular Position of masses

To determine the magnitude and directions of masses $m_M$ and $m_L$.

Let, $m_L$ be the balancing mass placed in plane L and $m_M$ be the balancing mass placed in plane M which are to be determined.

The data may be tabulated as shown.

| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/$\omega^2$ (m r) kg-m 4 | Distance from Ref. plane 'L' m 5 | Couple/$\omega^2$ (m r L) kg-$m^2$ 6 | Angle $\theta$ 7 |
|---|---|---|---|---|---|---|
| A | 50 | 0.025 | $m_A r_A$ = 1.25 | -0.3 | -0.375 | $\theta_A = 0^0$ |
| L (R.P.) | $m_L$ = ? | 0.10 | 0.1 $m_L$ | 0 | 0 | $\theta_L$ = ? |
| B | 40 | 0.025 | $m_B r_B$ = 1.00 | 0.3 | 0.3 | $\theta_B = 90^0$ |
| M | $m_M$ = ? | 0.10 | 0.1 $m_M$ | 0.6 | 0.06 $m_M$ | $\theta_M$ =? |
| C | 60 | 0.025 | $m_C r_C$ = 1.50 | 0.9 | 1.35 | $\theta_C = 210^0$ |

## Analytical Method:

**Step 1:**
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$\sum mrl \cos\theta + m_M r_M l_M \cos\theta_M = 0$ On substitution we get

$-0.375 \cos 0^0 + 0.3 \cos 90^0 + 0.06\, m_M \cos\theta_M + 1.35 \cos 210^0 = 0$ i.e. $-0.375 + 0 + 0.06\, m_M \cos\theta_M + (-1.16913) = 0$

$0.06\, m_M \cos\theta_M = 1.54413$

$m_M \cos\theta_M = \dfrac{1.54413}{\phantom{0.06}} = 25.74$ - - - - - (1)

Sum of the vertical components gives,

$\sum mrl \sin\theta + m_M r_M l_M \sin\theta_M = 0$ On substitution we get

$-0.375 \sin 0^0 + 0.3 \sin 90^0 + 0.06\, m_M \sin\theta_M + 1.35 \sin 210^0 = 0$ i.e. $0 + 0.3 + 0.06\, m_M \sin\theta_M + (-0.675) = 0$

$0.06\, m_M \sin\theta_M = 0.375$

$m_M \sin\theta_M = \dfrac{0.375}{0.06} = 6.25$ - - - - -(2)

Squaring and adding (1) and (2), we get

$$(m_M \cos\theta_M)^2 + (m_M \sin\theta_M)^2 = (25.74)^2 + (6.25)^2 = 701.61$$

i.e. $m_M^2 = 701.61$ and $m_M = 26.5$ kg Ans

Dividing (2) by (1), we get

$$\tan\theta_M = \frac{6.25}{25.74} \text{ and } \theta_M = 13.65^0 \quad \text{Ans}$$

**Step 2:**

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$\sum mr\cos\theta + m_L r_L \cos\theta_L + m_M r_M \cos\theta_M = 0$

On substitution we get

$1.25\cos 0^0 + 0.1m_L\cos\theta_L + 1.0\cos 90^0 + 2.649\cos 13.65^0 + 1.5\cos 210^0 = 0$

$1.25 + 0.1m_L\cos\theta_L + 0 + 2.5741 + (-1.299) = 0$

Therefore

$0.1m_L\cos\theta_L + 2.5251 = 0$

and $m_L\cos\theta_L = \frac{-2.5251}{0.1} = -25.251 - - - - - - - - - (3)$

Sum of the vertical components gives,

$\sum mr\sin\theta + m_L r_L \sin\theta_L + m_M r_M \sin\theta_M = 0$

On substitution we get

$1.25\sin 0^0 + 0.1m_L\sin\theta_L + 1.0\sin 90^0 + 2.649\sin 13.65^0 + 1.5\sin 210^0 = 0$

$0 + 0.1m_L\sin\theta_L + 1 + 0.6251 + (-0.75) = 0$

Therefore

$0.1m_L\sin\theta_L + 0.8751 = 0$

and $m_L\sin\theta_L = \frac{-0.8751}{0.1} = -8.751 - - - - - - - - - (4)$

Squaring and adding (3) and (4), we get

$$(m_L\cos\theta_L)^2 + (m_L\sin\theta_L)^2 = (-25.251)^2 + (-8.751)^2 = 714.193$$

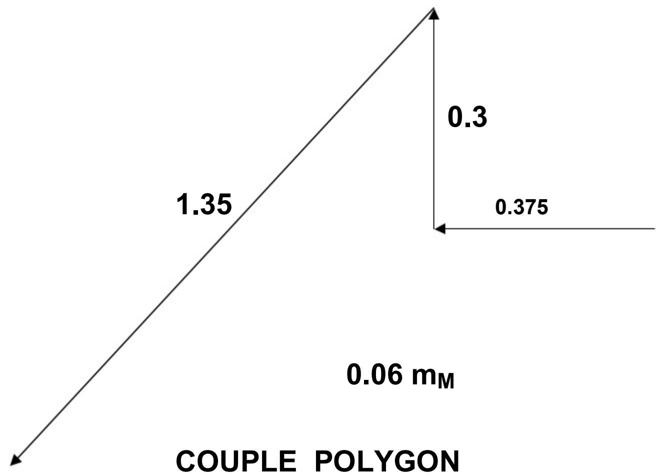
i.e. $m_L^2 = 714.193$ and $m_L = 26.72$ kg Ans

Dividing (4) by (3), we get

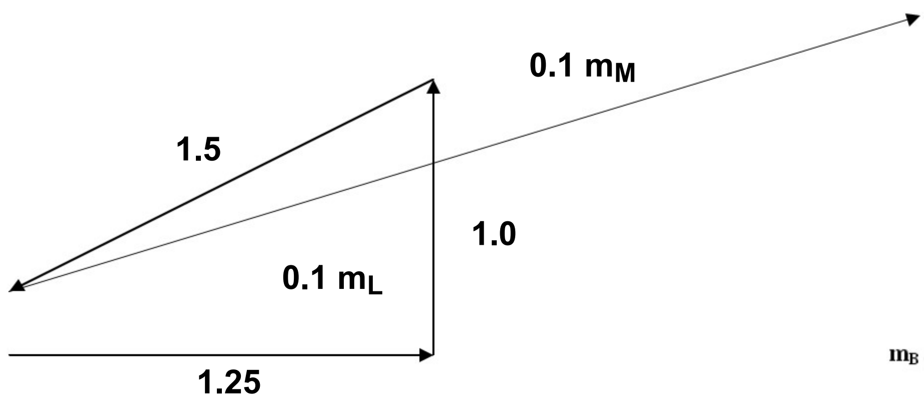
$$\tan\theta_L = \frac{-8.751}{-25.251} \quad \text{and } \theta_L = 19.11^0 \text{ Ans}$$

The balancing mass $m_L$ is at an angle $19.11^0 + 180^0 = 199.11^0$ measured in counter clockwise direction.

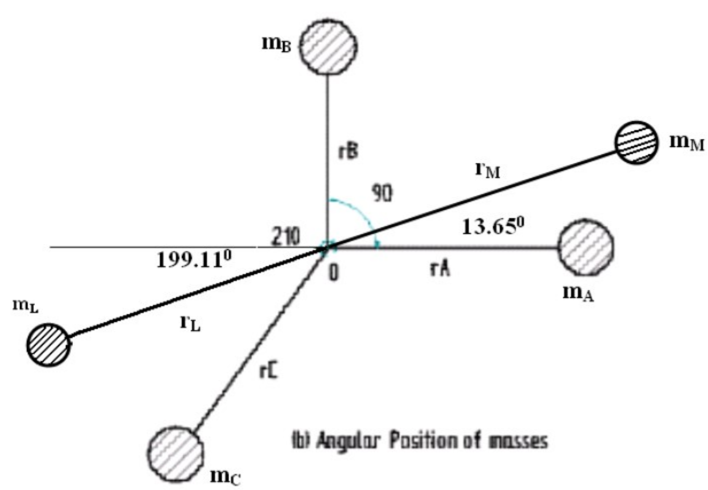
## Graphical Method:




COUPLE POLYGON




FORCE POLYGON




(b) Angular Position of masses

**Problem 8:**

Four masses A, B, C and D are completely balanced. Masses C and D make angles of $90^0$ and $210^0$ respectively with B in the same sense. The planes containing B and C are 300 mm apart. Masses A, B, C and D can be assumed to be concentrated at radii of 360 mm, 480 mm, 240 mm and 300 mm respectively. The masses B, C and D are 15 kg, 25 kg and 20 kg respectively. Determine i) mass A and its angular position ii) position of planes A and D.

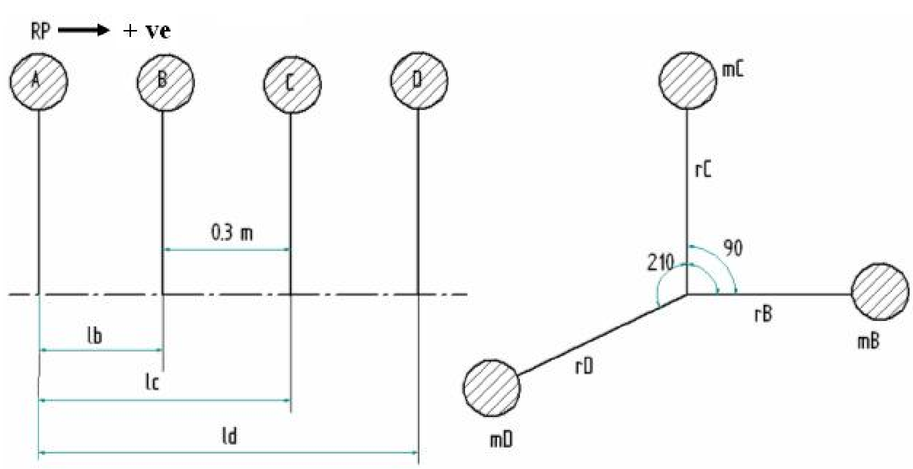
**Solution:** **Analytical Method**

**Step 1:**

Draw the space diagram or angular position of the masses. Since the angular position of the masses C and D are given with respect to mass B, take the angular position of mass B as $\theta_B = 0^0$.

Tabulate the given data as shown.

| Plane 1 | Mass (m) kg 2 | Radius (r) m 3 | Centrifugal force/$\omega^2$ (m r) kg-m 4 | Distance from Ref. plane 'A' m 5 | Couple/ $\omega^2$ (m r L) kg-m$^2$ 6 | Angle $\theta$ 7 |
|---|---|---|---|---|---|---|
| A (R.P.) | $m_A$= ? | 0.36 | $m_A r_A = 0.36\ m_A$ | 0 | 0 | $\theta_A$ =? |
| B | 15 | 0.48 | $m_B r_B = 7.2$ | $l_B$ = ? | 7.2 $l_B$ | $\theta_B$ =0 |
| C | 25 | 0.24 | $m_C r_C = 6.0$ | $l_C$ = ? | 6.0 $l_C$ | $\theta_c = 90^0$ |
| D | 20 | 0.30 | $m_D r_D = 6.0$ | $l_D$ = ? | 6.0 $l_D$ | $\theta_D = 210^0$ |




(a) Position of planes of masses (Assumed) (b) Angular position of masses

**Step 2:**

Mass $m_A$ be the balancing mass placed in plane A which is to be determined along with its angular position.

Refer column 4 of the table. Since $m_A$ is to be determined ( which is the only unknown) ,resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D = 0$$

**On substitution we get**

$$0.36\, m_A \cos\theta_A + 7.2\cos 0^0 + 6.0\cos 90^0 + 6.0\cos 210^0 = 0$$

**Therefore**

$$0.36\, m_A \cos\theta_A = -2.004 \text{ ---------- } (1)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D = 0$$

On substitution we get

$$0.36\, m_A \sin\theta_A + 7.2\sin 0^0 + 6.0\sin 90^0 + 6.0\sin 210^0 = 0$$

Therefore

$$0.36\, m_A \sin\theta_A = -3.0 \text{ ---------- } (2)$$

Squaring and adding (1) and (2), we get

$$0.36^2 (m_A)^2 = (-2.004)^2 + (-3.0)^2 = 13.016$$

$$m_A = \sqrt{\frac{13.016}{}} = 10.02 \quad \text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_A = \frac{-3.0}{-2.004}$$ and Resutltant makes an angle = $56.26^0$

The balancing mass A makes an angle of $\theta_A = 236.26^0$ Ans

**Step 3:**

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta = m_A r_A l_A \cos\theta_A + m_B r_B l_B \cos\theta_B + m_C r_C l_C \cos\theta_C + m_D r_D l_D \cos\theta_D = 0$$

**On substitution we get**

$$0 + 7.2\, l_B \cos 0^0 + 6.0\, l_C \cos 90^0 + 6.0\, l_D \cos 210^0 = 0$$

$$7.2\, l_B - 5.1962\, l_D = 0 \text{ - - - - - - - - - - - - } (3)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta = m_A r_A l_A \sin\theta_A + m_B r_B l_B \sin\theta_B + m_C r_C l_C \sin\theta_C + m_D r_D l_D \sin\theta_D = 0$$

**On substitution we get**

$$0 + 7.2\, l_B \sin 0^0 + 6.0\, l_C \sin 90^0 + 6.0\, l_D \sin 210^0 = 0$$

**But from figure we have,** $l_C = l_B + 0.3$

On substituting this in equation (4), we get

$$6.0\,(l_B + 0.3) - 3\, l_D = 0$$

$$\text{i.e. } 6.0\, l_B - 3\, l_D = 1.8 \text{ - - - - - - - - - - -}(5)$$

**Thus we have two equations ( 3) and (5), and two unknowns** $l_B$, $l_D$

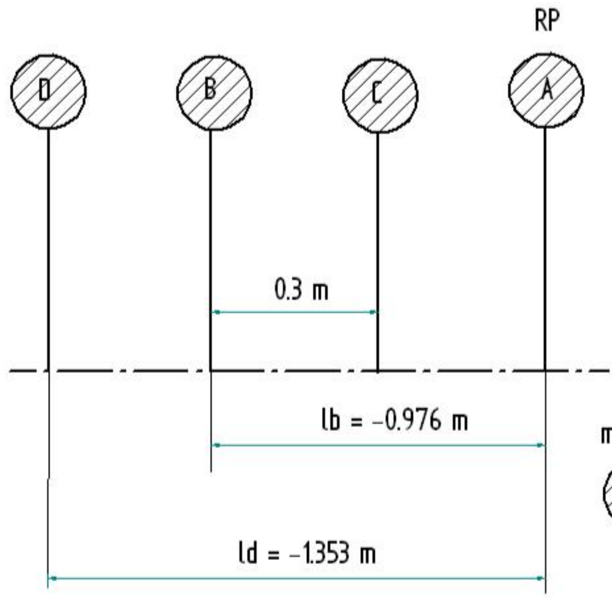
$$7.2\, l_B - 5.1962\, l_D = 0 \text{ - - - - - - - - - - - - } (3)$$

$$6.0\, l_B - 3\, l_D = 1.8 \text{ - - - - - - - - - - -}(5)$$

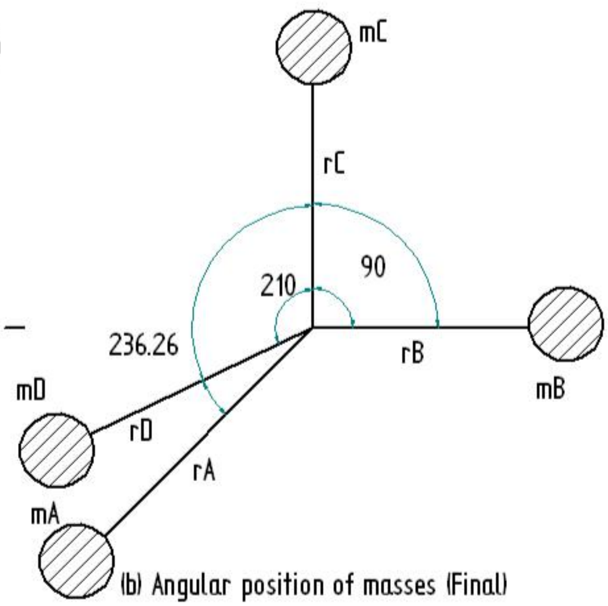
**On solving the equations, we get**

$l_D = -1.353$ m and $l_B = -0.976$ m

As per the position of planes of masses assumed the distances shown are positive (+ ve ) from the reference plane A. But the calculated values of distances $l_B$ and $l_D$ are negative. The corrected positions of planes of masses is shown below.




(a) Position of planes of masses (Corrected)




(b) Angular position of masses (Final)

# BALANCING OF RECIPROCATING MASSES

## SLIDER CRANK MECHANISM:

### PRIMARY AND SECONDARY ACCELERATING FORCE:

Acceleration of the reciprocating mass of a slider-crank mechanism is given by,

$a_p$ = **Acceleration of piston**

$$= r\omega^2 \cos\theta + \frac{\cos 2\theta}{n} \text{ ————————(1)}$$

**Where** $n = \frac{l}{r}$

And, the force required to accelerate the mass 'm' is

$$F_i = mr\omega^2 \cos\theta + \frac{\cos 2\theta}{n}$$

$$= mr\omega^2 \cos\theta + mr\omega^2 \frac{\cos 2\theta}{n} \text{ ————————(2)}$$

The first term of the equation (2) , i.e. **mr** $\omega^2$ **cos**$\theta$ is called **primary accelerating force** the second term **mr** $\omega^2 \frac{\cos 2\theta}{n}$ is called the **secondary accelerating force.**

Maximum value of primary accelerating force is **mr** $\omega^2$

And Maximum value of secondary accelerating force is $\frac{mr\,\omega^2}{n}$

Generally, 'n' value is much greater than one; the secondary force is small compared to primary force and can be safely neglected for slow speed engines.

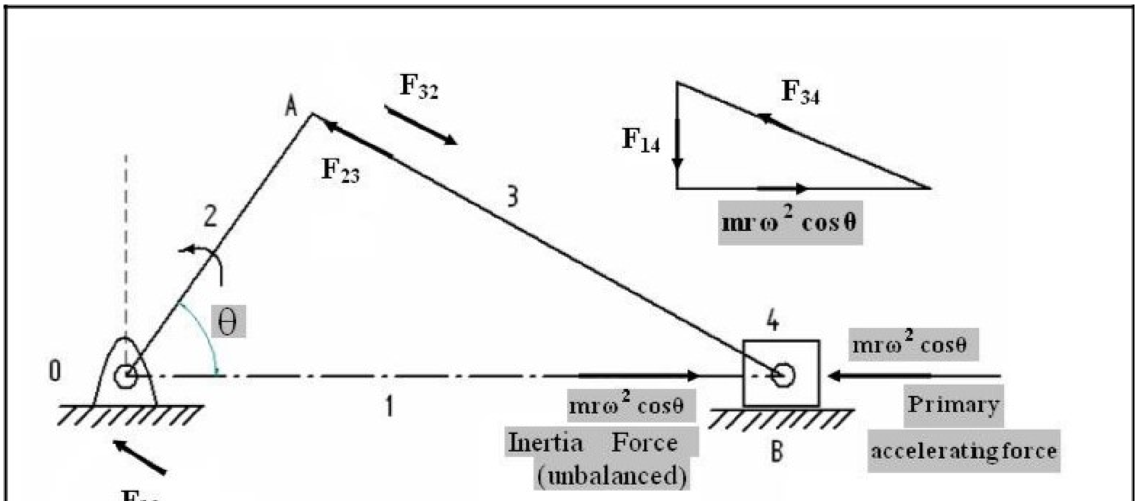


Figure (a)

In Fig (a), the inertia force due to primary accelerating force is shown.

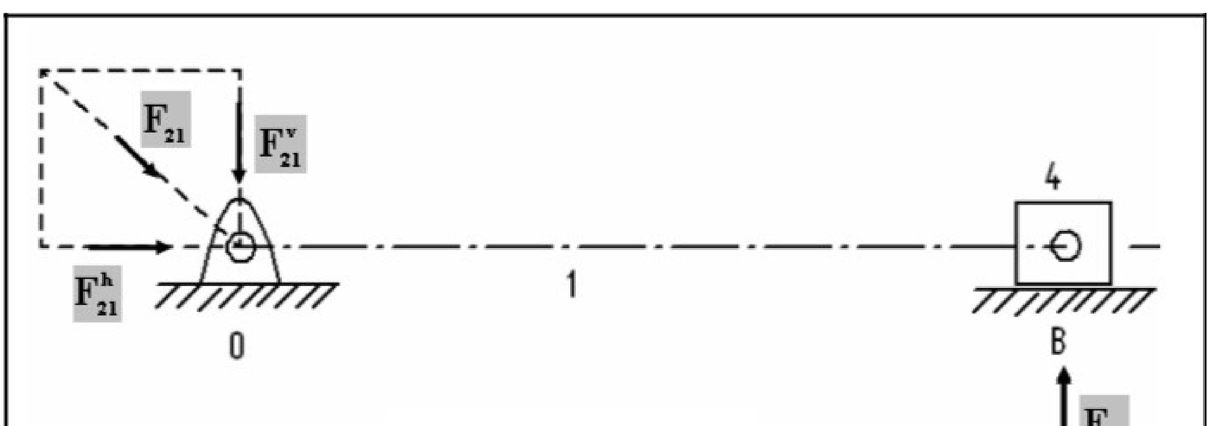


Figure (b)

In Fig (b), the forces acting on the engine frame due to inertia force are shown.

At 'O' the force exerted by the crankshaft on the main bearings has two components, horizontal $\mathbf{F}_{21}{}^{\mathbf{h}}$ and vertical $\mathbf{F}_{21}{}^{\mathbf{v}}$.

$\mathbf{F}_{21}{}^{\mathbf{h}}$ is an horizontal force, which is an **unbalanced shaking force.**

$\mathbf{F}_{21}{}^{\mathbf{v}}$ and $\mathbf{F}_{41}{}^{\mathbf{v}}$ balance each other but form an **unbalanced shaking couple**.

The magnitude and direction of these unbalanced force and couple go on changing with angle θ. The shaking force produces linear vibrations of the frame in horizontal direction, whereas the shaking couple produces an oscillating vibration.
The shaking force $\mathbf{F}_{21}{}^{\mathbf{h}}$ is the only unbalanced force which may hamper the smooth running of the engine and effort is made to balance the same.
However it is not at all possible to balance it completely and only some modifications can be carried out.

**BALANCING OF THE SHAKING FORCE:**

Shaking force is being balanced by adding a rotating counter mass at radius 'r' directly opposite the crank. This provides only a partial balance. This counter mass is in addition to the mass used to balance the rotating unbalance due to the mass at the crank pin. This is shown in figure (c).

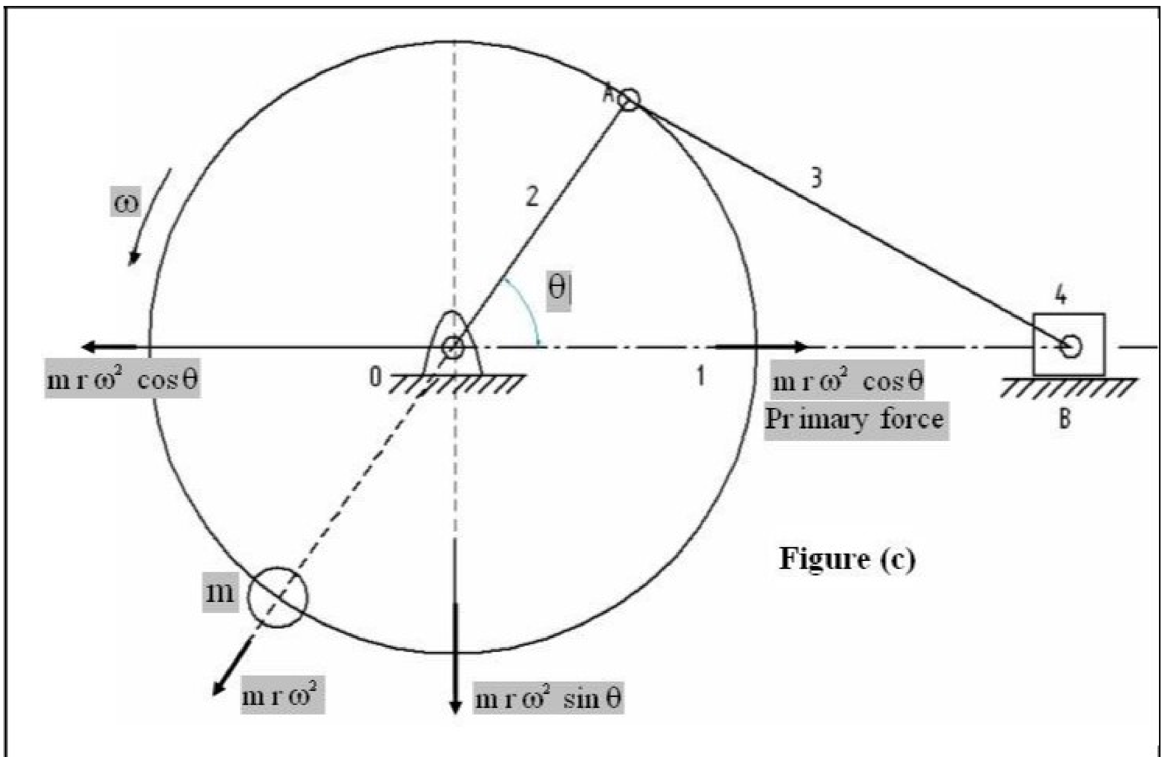



Figure (c)

The horizontal component of the centrifugal force due to the balancing mass is $mr\,\omega^2 \cos\theta$ and this is in the line of stroke. This component neutralizes the unbalanced reciprocating force. But the rotating mass also has a component $mr\,\omega^2 \sin\theta$ perpendicular to the line of stroke which remains unbalanced. The unbalanced force is zero at $\theta = 0^0$ or $180^0$ and maximum at the middle of the stroke i.e. $\theta = 90^0$. The magnitude or the maximum value of the unbalanced force remains the same i.e. equal to $mr\,\omega^2$. Thus instead of sliding to and fro on its mounting, the mechanism tends to jump up and down.

To minimize the effect of the unbalance force a compromise is, usually made, is $\frac{2}{3}$ of the reciprocating mass is balanced or a value between $\frac{1}{2}$ to $\frac{3}{4}$.

If 'c' is the fraction of the reciprocating mass, then

The primary force balanced by the mass = $c\,mr\,\omega^2 \cos\theta$

and

The primary force unbalanced by the mass = $(1 - c)\,mr\,\omega^2 \cos\theta$

Vertical component of centrifuga l force which remains unbalanced

4. $c\,m\,r\,\omega^2 \sin\theta$

In reciprocating engines, unbalance forces in the direction of the line of stroke are more dangerous than the forces perpendicular to the line of stroke.

Resultant unbalanced force at any instant

5. $\sqrt{[(1 - c)mr\,\omega^2 \cos\theta]^2 + [c\,mr\,\omega^2 \sin\theta]^2}$

The resultant unbalanced force is minimum when, $c = \frac{1}{2}$

This method is just equivalent to as if a revolving mass at the crankpin is completely balanced by providing a counter mass at the same radius diametrically opposite to the crank. Thus if $m_P$ is the mass at the crankpin and 'c' is the fraction of the reciprocating mass 'm' to be balanced , the mass at the crankpin may be considered as $cm + m_P$ which is to be completely balanced.

**Problem 1:**

A single –cylinder reciprocating engine has a reciprocating mass of 60 kg. The crank rotates at 60 rpm and the stroke is 320 mm. The mass of the revolving parts at 160 mm radius is 40 kg. If two-thirds of the reciprocating parts and the whole of the revolving parts are to be balanced, determine the, (i) balance mass required at a radius of 350 mm and (ii) unbalanced force when the crank has turned $50^0$ from the top-dead centre.

Solution:

Given : m = mass of the reciprocating parts = 60 kg N = 60 rpm, L = length of the stroke = 320 mm $m_P$ = 40 kg, $c = \frac{2}{3}$, $r_c$ =350 mm

(i) Balance mass required at a radius of 350 mm

We have, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 60}{60} = 2\pi$ rad/s

$r = \frac{L}{2} = \frac{320}{2} = 160$ mm

Mass to be balanced at the crank pin = M

$M = c\,m + m_P = \frac{2}{3} \times 60 + 40 = 80$ kg

and $m_c r_c = M r$ therefore $m_c = \frac{M r}{r_c}$

i.e. $m_c = \frac{80 \times 160}{350} = 36.57$ kg

(ii) Unbalanced force when the crank has turned $50^0$ from the top-dead centre.

Unbalanced force at $\theta = 50^0$

$$= \sqrt{\left[(1 - c)mr\,\omega^2 \cos\theta\right]^2 + \left[c\,mr\,\omega^2 \sin\theta\right]^2}$$

$$= \sqrt{\left[\left(1 - \frac{2}{3}\right) \times 60 \times 0.16 \times (2\pi)^2 \cos 50^0\right]^2 + \left[\frac{2}{3} \times 60 \times 0.16 \times (2\pi)^2 \sin 50^0\right]^2}$$

3. 209.9 N

**Problem 2:**

The following data relate to a single cylinder reciprocating engine:
Mass of reciprocating parts = 40 kg
Mass of revolving parts = 30 kg at crank radius
Speed = 150 rpm, Stroke = 350 mm.
If 60 % of the reciprocating parts and all the revolving parts are to be balanced, determine the,

c) balance mass required at a radius of 320 mm and (ii) unbalanced force when the crank has turned $45^0$ from the top-dead centre.

Solution:

Given : m = mass of the reciprocating parts = 40 kg
$m_P$ = 30 kg , N = 150 rpm, L = length of the stroke = 350 mm
= 60 % , $r_c$ =320 m m

= Balance mass required at a radius of 350 mm

We have,

$$\omega = \frac{2 \pi N}{60} = \frac{2 \pi \times 150}{60} = 15.7 \text{ rad/s}$$

$$r = \frac{L}{2} = \frac{350}{2} = 175 \text{ mm}$$

Mass to be balanced at the crank pin = M

$$M = c\,m + m_P = 0.60 \times 40 + 30 = 54 \text{ kg}$$

and

$$m_c r_c = M r \quad \text{therefore} \quad m_c = \frac{M r}{r_c}$$

$$\text{i.e. } m_c = \frac{54 \times 175}{320} = 29.53 \text{ kg}$$

(ii) Unbalanced force when the crank has turned $45^0$ from the top-dead centre.

Unbalance d force at $\theta = 45^0$

$$= \sqrt{[(1 - c) m r \omega^2 \cos\theta]^2 + [c\, m r \omega^2 \sin\theta]^2}$$

$$= \sqrt{[(1 - 0.60) \times 40 \times 0.175 \times (15.7)^2 \cos 45^0]^2 + [0.60 \times 40 \times 0.175 \times (15.7)^2 \sin 45^0]^2}$$

= 880.7 N

**SECONDARY BALANCING:**

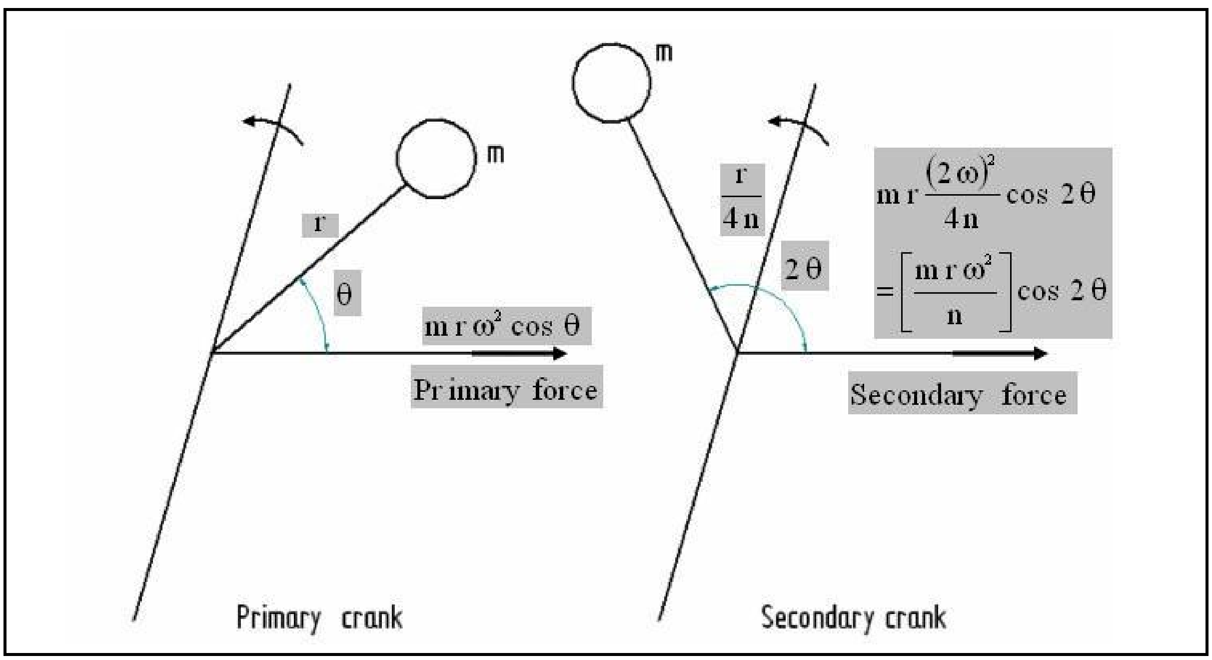
Secondary acceleration force is equal to $mr\,\omega^2 \dfrac{\cos 2\theta}{n}$ ------(1)

Its frequency is twice that of the primary force and the magnitude $\dfrac{1}{n}$ times the magnitude of the primary force.

The secondary force is also equal to $mr\,(2\omega)^2 \dfrac{\cos 2\theta}{4n}$ ------(2)

Consider, two cranks of an engine, one actual one and the other imaginary with the following specifications.

| | Actual | Imaginary |
|---|---|---|
| Angular velocity | $\omega$ | $2\omega$ |
| Length of crank | $r$ | $\dfrac{r}{4n}$ |
| Mass at the crank pin | m | m |




Thus, when the actual crank has turned through an angle $\theta = \omega t$, the imaginary crank would have turned an angle $2\theta = 2\omega t$

Centrifugal force induced in the imaginary crank = $\frac{m r (2\omega)^2}{4n}$

Component of this force along the line of stroke is = $\frac{m r (2\omega)^2}{4n} \cos 2\theta$

Thus the effect of the secondary force is equivalent to an imaginary crank of length $\frac{r}{4n}$ rotating at double the angular velocity, i.e. twice of the engine speed. The imaginary crank coincides with the actual at inner top-dead centre. At other times, it makes an angle with the line of stroke equal to twice that of the engine crank.

The secondary couple about a reference plane is given by the multiplication of the secondary force with the distance ' l ' of the plane from the reference plane.

**COMPLETE BALANCING OF RECIPROCATING PARTS**

Conditions to be fulfilled:

3. Primary forces must balance i.e., primary force polygon is enclosed.
4. Primary couples must balance i.e., primary couple polygon is enclosed.
5. Secondary forces must balance i.e., secondary force polygon is enclosed.
6. Secondary couples must balance i.e., secondary couple polygon is enclosed.

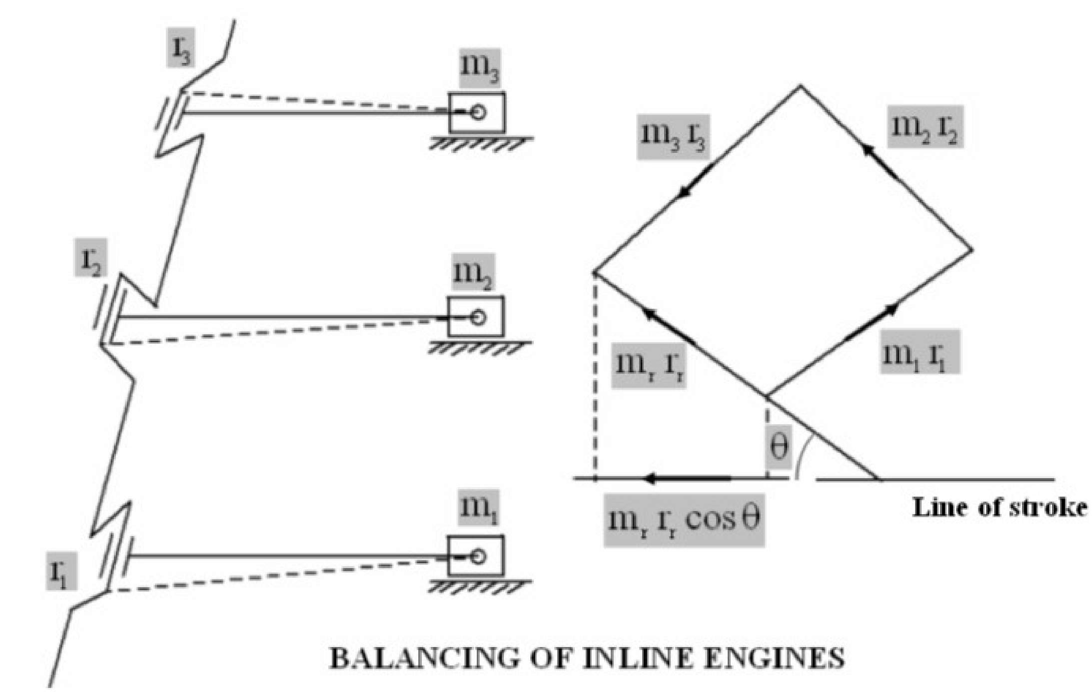
Usually, it is not possible to satisfy all the above conditions fully for multi-cylinder engine. Mostly some unbalanced force or couple would exist in the reciprocating engines.

**BALANCING OF INLINE ENGINES:**

An in-line engine is one wherein all the cylinders are arranged in a single line, one behind the other. Many of the passenger cars such as Maruti 800, Zen, Santro, Honda-city, Honda CR-V, Toyota corolla are the examples having four cinder in-line engines.

In a reciprocating engine, the reciprocating mass is transferred to the crankpin; the axial component of the resulting centrifugal force parallel to the axis of the cylinder is the primary unbalanced force.

Consider a shaft consisting of three equal cranks asymmetrically spaced. The crankpins carry equivalent of three unequal reciprocating masses, then



BALANCING OF INLINE ENGINES

Primary force = $\sum m\, r\, \omega^2 \cos\theta$ ------------- (1)

Primary couple = $\sum m\, r\, \omega^2\, l \cos\theta$ ------------- (2)

Secondary force = $\sum m\, r \frac{(2\omega)^2}{4n} \cos 2\theta$ -------------(3)

And Secondary couple = $\sum m\, r \frac{(2\omega)^2}{4\,n}\, l \cos 2\theta$

$$= \sum m\, r \frac{\omega^2}{n}\, l \cos 2\theta \text{ ------------- (4)}$$

**GRAPHICAL SOLUTION:**

To solve the above equations graphically, first draw the $\sum m\, r \cos\theta$ polygon ( $\omega^2$ is common to all forces). Then the axial component of the resultant forces

multiplied by $\omega^2$ provides the primary unbalanced force on the system at that **($F_r \cos\theta$)** moment.

This unbalanced force is zero when $\theta = 90^0$ and a maximum when $\theta = 0^0$ .

If the force polygon encloses, the resultant as well as the axial component will always be zero and the system will be in **primary balance**.
Then,

$$\sum F_{Ph} = 0 \text{ and } \sum F_{PV} = 0$$

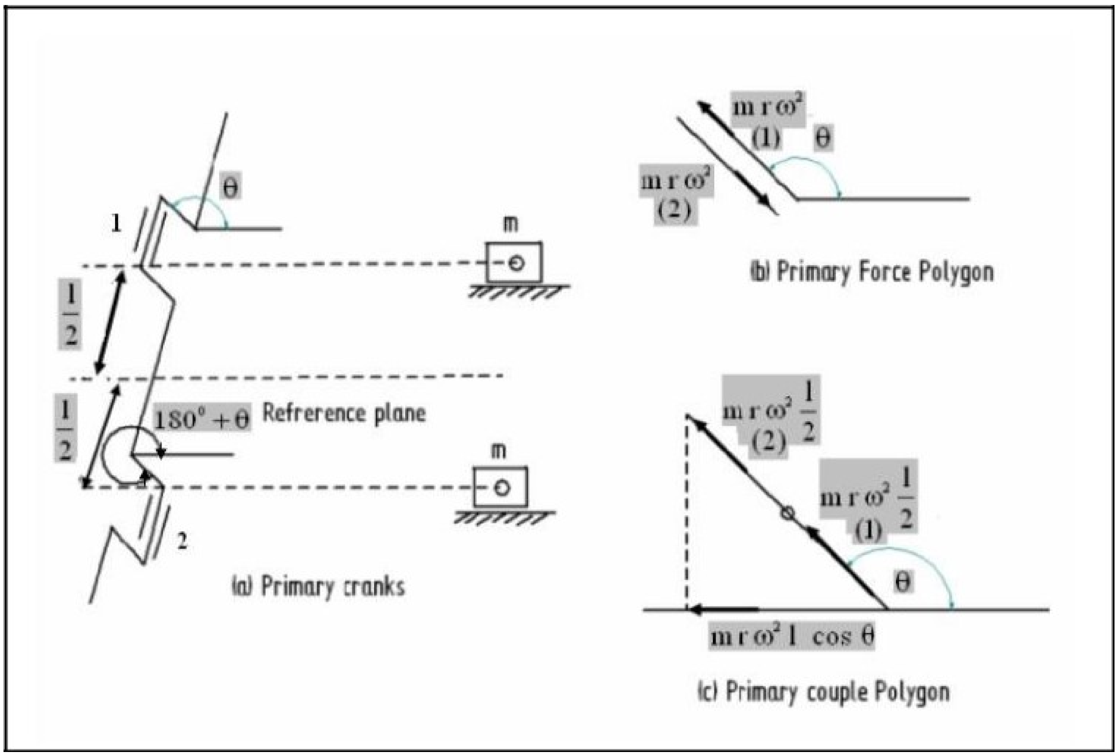
To find the secondary unbalance force, first find the positions of the imaginary secondary cranks. Then transfer the reciprocating masses and multiply the same by $\frac{(2\omega)^2}{4n}$ or $\frac{\omega^2}{n}$ to get the secondary force.

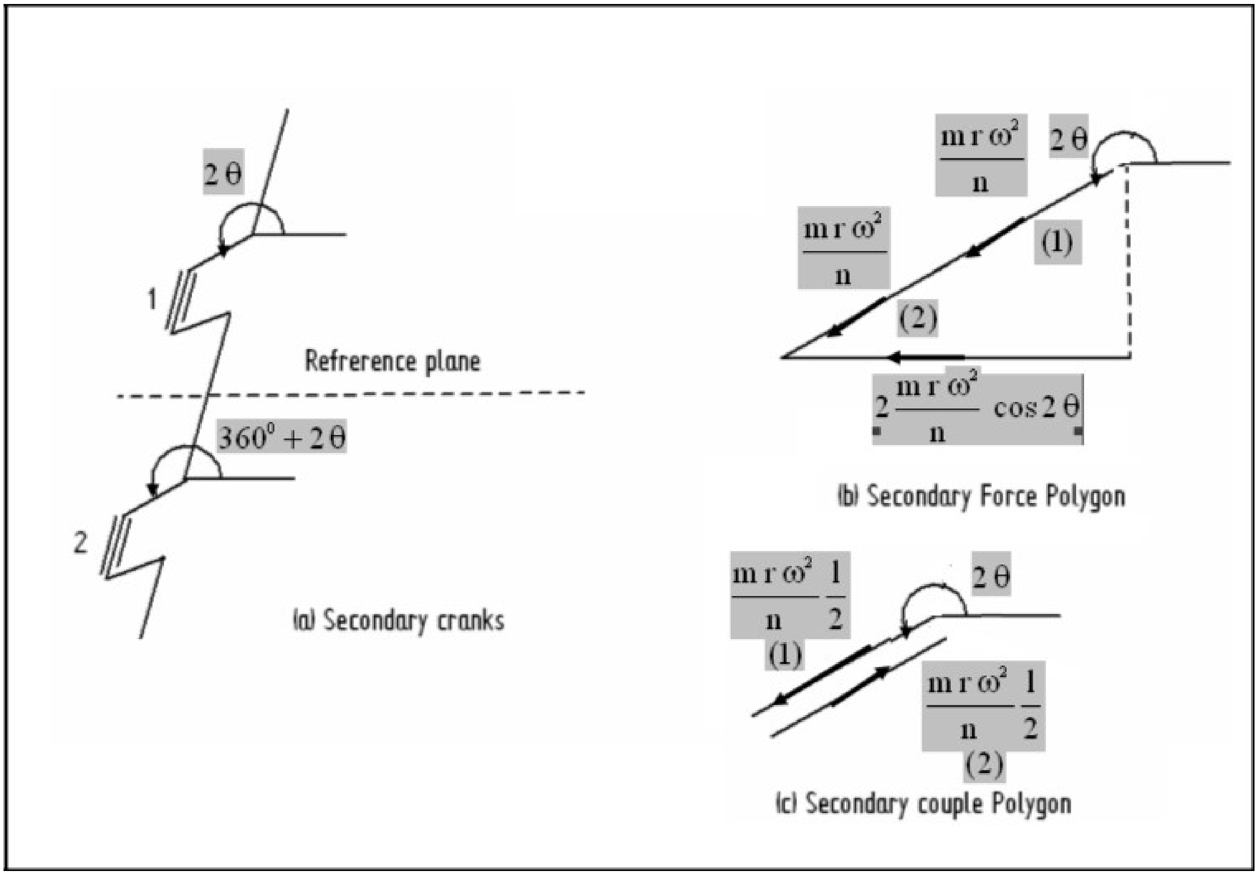
In the same way primary and secondary couple ( m r l ) polygon can be drawn for primary and secondary couples.

**Case 1:**

**IN-LINE TWO-CYLINDER ENGINE**

Two-cylinder engine, cranks are $180^0$ apart and have equal reciprocating masses.






Taking a plane through the **centre line** as the reference plane,

Primary force $= m r \omega^2 [\cos\theta + \cos(180 + \theta)] = 0$

Primary couple $= m r \omega^2 \frac{l}{2} \cos\theta + -\frac{l}{2}\cos(180 + \theta) = m r \omega^2 l \cos\theta$

Maximum values are $m r \omega^2 l$ at $\theta = 0^0$ and $180^0$

Secondary force $= \frac{m r \omega^2}{n}[\cos 2\theta + \cos(360 + 2\theta)] = \frac{2 m r \omega^2}{n} \cos 2\theta$

Maximum values are $\frac{2 m r \omega^2}{n}$ when $2\theta = 0^0, 180^0, 360^0$ and $540^0$

or $\theta = 0^0, 90^0, 180^0$ and $270^0$

Secondary couple $= \frac{m r \omega^2}{n} \left[ \frac{l}{2} \cos 2\theta + \frac{l}{2} \cos(360 + 2\theta) \right] = 0$

ANALYTICAL METHOD OF FINDING PRIMARY FORCES AND COUPLES

= First the positions of the cranks have to be taken in terms of $\theta_0$

= The maximum values of these forces and couples vary instant to instant and are equal to the values as given by the equivalent rotating masses at the crank pin.

If a particular position of the crank shaft is considered, the above expressions may not give the maximum values.
For example, the maximum value of primary couple is $m r \omega^2 l$ and this value is obtained at crank positions $0^0$ and $180^0$. However, if the crank positions are assumed at $90^0$ and $270^0$, the values obtained will be zero.

= If any particular position of the crank shaft is considered, then both X and Y components of the force and couple can be taken to find the maximum values.

For example, if the crank positions considered as $120^0$ and $300^0$, the primary couple can be obtained as

$$X - \text{component} = m r \omega^2 \left[ \frac{l}{2} \cos 120^0 + -\frac{l}{2} \cos(180^0 + 120^0) \right]$$

3. $-\frac{1}{2} m r \omega^2 l$

$$Y - \text{component} = m r \omega^2 \left[ \frac{l}{2} \sin 120^0 + -\frac{l}{2} \sin(180^0 + 120^0) \right]$$

$$= \frac{\sqrt{3}}{2} m r \omega^2 l$$

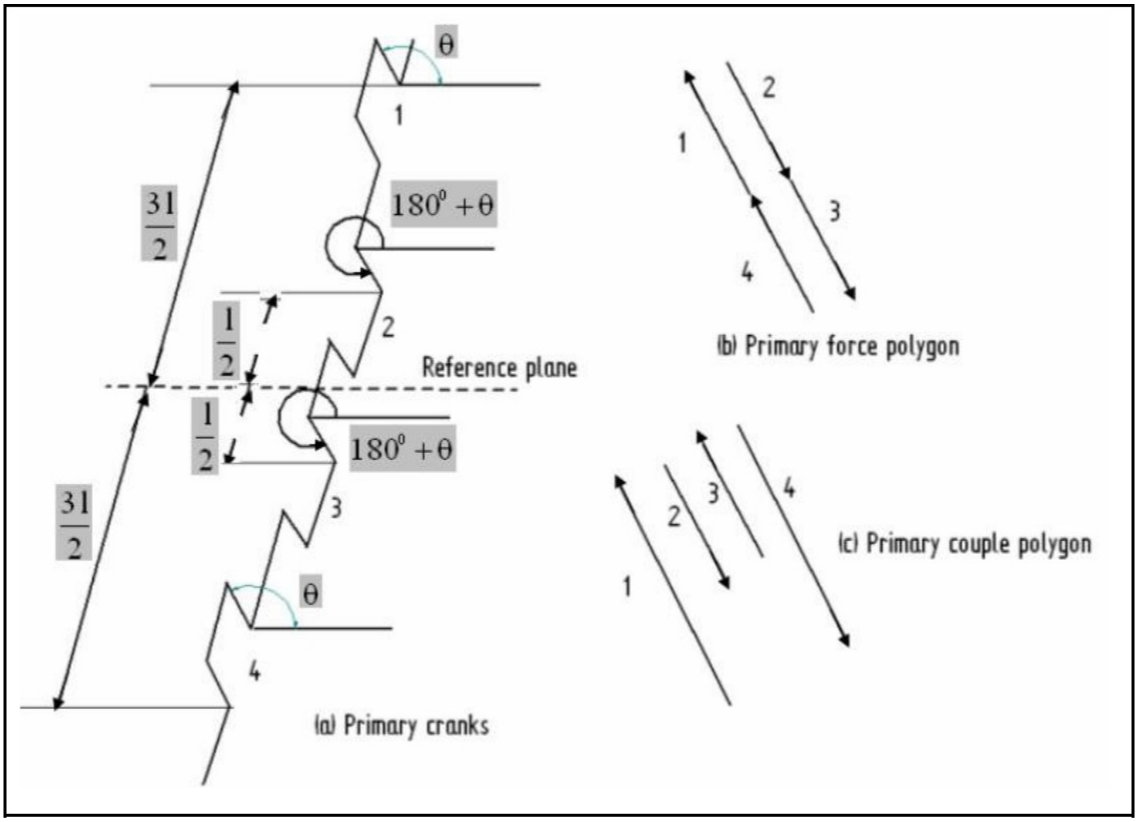
Therefore, Primary couple $= \sqrt{\left(\frac{1}{2} m r \omega^2 l\right)^2 + \left(\frac{\sqrt{3}}{2} m r \omega^2 l\right)^2}$

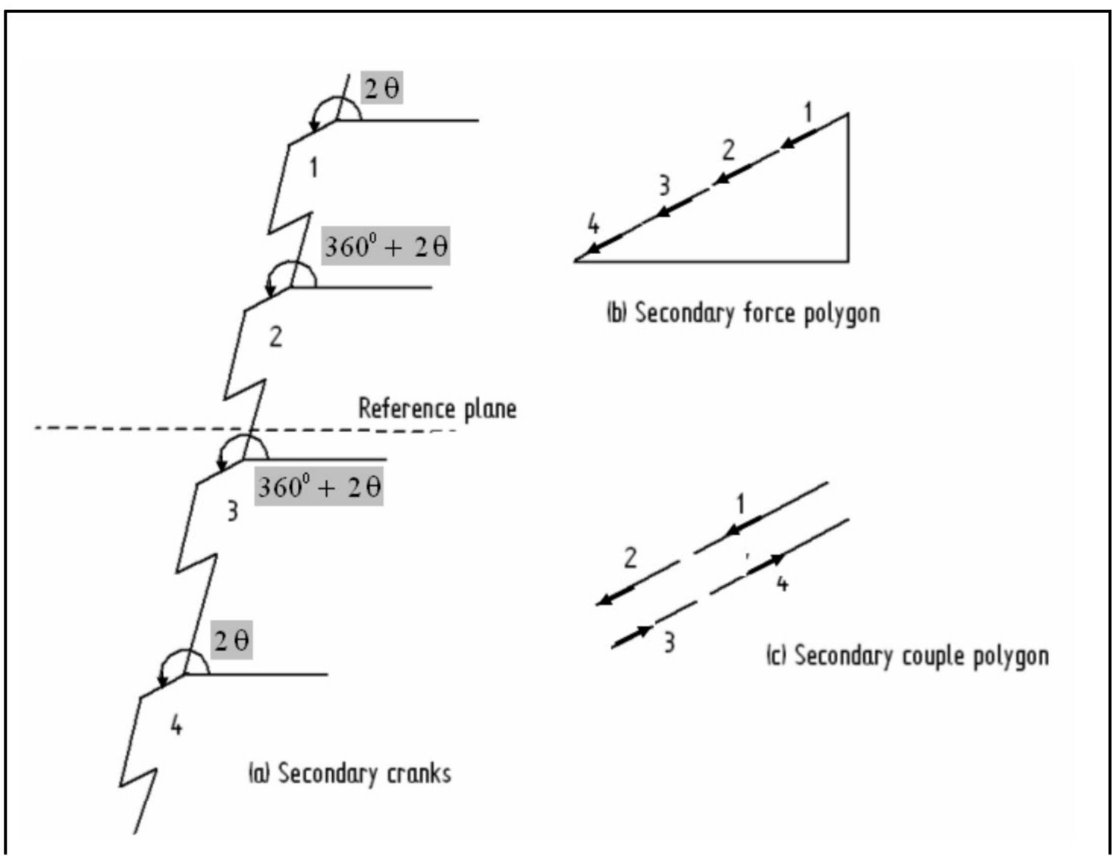
$= m r \omega^2 l$

**Case 2:**

## IN-LINE FOUR-CYLINDER FOUR-STROKE ENGINE

This engine has tow outer as well as inner cranks (throws) in line. The inner throws are at $180^0$ to the outer throws. Thus the angular positions for the cranks are $\theta^0$ for the first, $180^0 + \theta^0$ for the second, $180^0 + \theta^0$ for the third and $\theta^0$ for the fourth.



**FINDING PRIMARY FORCES, PRIMARY COUPLES, SECONDARY FORCES AND SECONDARY COUPLES:**

Choose a plane passing through the middle bearing about which the arrangement is symmetrical as the reference plane.

$$\text{Primary force} = m r \omega^2 \left[\cos\theta + \cos(180^0+\theta) + \cos(180^0+\theta) + \cos\theta\right] = 0$$

$$\text{Primary couple} = m r \omega^2 \left[\frac{3l}{2}\cos\theta + \frac{l}{2}\cos(180^0+\theta) + -\frac{l}{2}\cos(180^0+\theta) + -\frac{3l}{2}\cos\theta\right] = 0$$

$$\text{Secondary force} = \frac{m r \omega^2}{n}\left[\cos 2\theta + \cos(360^0+2\theta) + \cos(360^0+2\theta) + \cos 2\theta\right]$$

$$\rho\ \frac{4 m r \omega^2}{n}\cos 2\theta$$

$$\text{Maximum value} = \frac{m r \omega^2}{n}$$

at $2\theta = 0^0, 180^0, 360^0$ and $540^0$ or

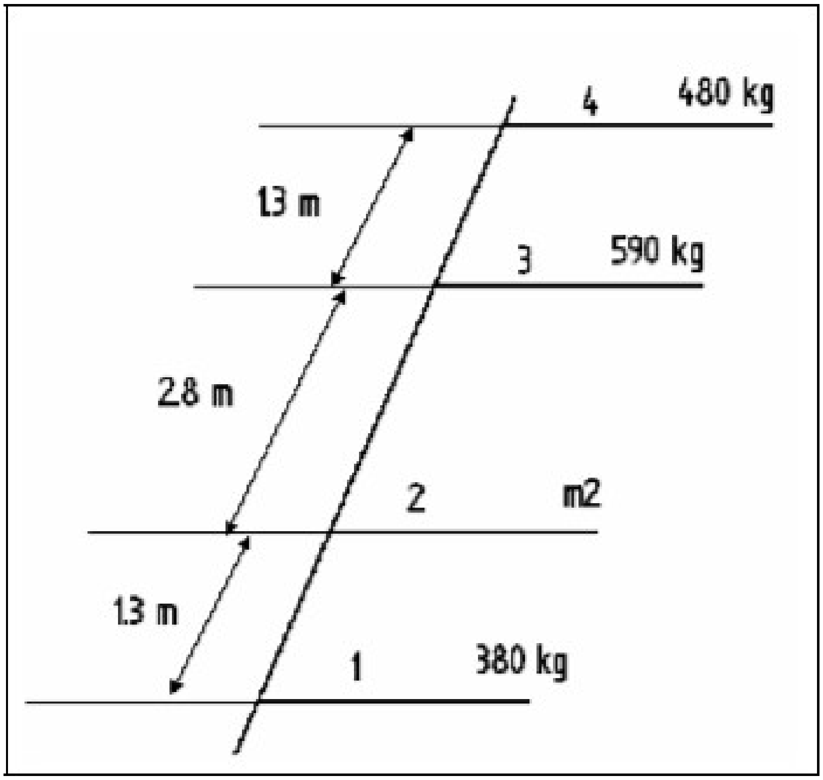
$\theta = 0^0, 90^0, 180^0$ and $270^0$

$$\text{Secondary couple} = \frac{m r \omega^2}{n}\left[\frac{3l}{2}\cos 2\theta + \frac{l}{2}\cos(360^0+2\theta) + -\frac{l}{2}\cos(360^0+2\theta) + -\frac{3l}{2}\cos 2\theta\right] = 0$$

**Thus the engine is not balanced in secondary forces.**

**Problem 1:**

A four-cylinder oil engine is in complete primary balance. The arrangement of the reciprocating masses in different planes is as shown in figure. The stroke of each piston is 2 r mm. Determine the reciprocating mass of the cylinder 2 and the relative crank position.




Solution:

Given :

$m_1 = 380$ kg, $m_2 = ?$ , $m_3 = 590$ kg, $m_4 = 480$ kg

crank length = $\frac{L}{2} = \frac{2r}{2} = r$

| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r ) kg m | Distance from Ref plane '2' m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 380 | r | 380 r | -1.3 | -494 r |
| 2(RP) | $m_2$ | r | $m_2$ r | 0 | 0 |
| 3 | 590 | r | 590 r | 2.8 | 1652 r |
| 4 | 480 | r | 480 r | 4.1 | 1968 r |

**Analytical Method:**

Choose plane 2 as the reference plane and $\theta_3 = 0^0$ .

.

**Step 1:**

Resolve the couples into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$$-494 \, \mathbf{r} \cos \theta_1 + 1652 \, \mathbf{r}\cos 0^0 + 1968 \, \mathbf{r}\cos \theta_4 = 0$$

**i.e.,** $+ 494 \cos \theta_1 = 1652 + 1968 \cos \theta_4$ – – – – – – – – – – **(1)**

Sum of the vertical components gives

$$- 494 \, \mathbf{r} \sin \theta_1 + 1652 \mathbf{r} \sin 0^0 + 1968 \, \mathbf{r} \sin \theta_4 = 0$$

**i.e.,** $494 \sin \theta_1 = 1968 \sin \theta_4$ – – – – – – – – – – **(2)**

Squaring and adding (1) and (2), we get

$$(494)^2 = (1652 + 1968 \cos \theta_4)^2 + (1968 \sin \theta_4)^2$$

**i.e.,**

$(494)^2 = (1652)^2 + 2 \times 1652 \times 1968 \cos \theta_4 + (1968 \cos \theta_4)^2 + (1968 \sin \theta_4)^2$ On solving w e get,

$\cos \theta_4 = -0.978$ and $\theta_4 = 167.9^0$ or $192.1^0$

Choosing one value, say $\theta_4 = 167.9^0$

Dividing (2) by (1), we get

$$\tan \theta_1 = \frac{1968 \sin(167.9^0)}{1652 + 1968 \cos(167.9^0)} = \frac{+412.53}{-272.28} = -1.515$$

i.e., $\theta_1 = 123.4^0$

Step 2:

Resolve the forces into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$$380 \, \mathbf{r} \cos(123.4^0) + \mathbf{m}_2 \mathbf{r} \cos \theta_2 + 590 \, \mathbf{r}\cos 0^0 + 480 \mathbf{r}\cos(167.9^0) = 0$$

**or** $\mathbf{m}_2 \cos \theta_2 = 88.5$ – – – – – – – – – – – – – – **(3)**

Sum of the vertical components gives

$$380\, \mathbf{r}\, \mathbf{sin}(123.4^0) + \mathbf{m}_2\, \mathbf{r}\, \mathbf{sin}\, \theta_2 + 590\, \mathbf{r}\, \mathbf{sin} 0^0 + 480 \mathbf{rsin}(167.9^0) = 0$$

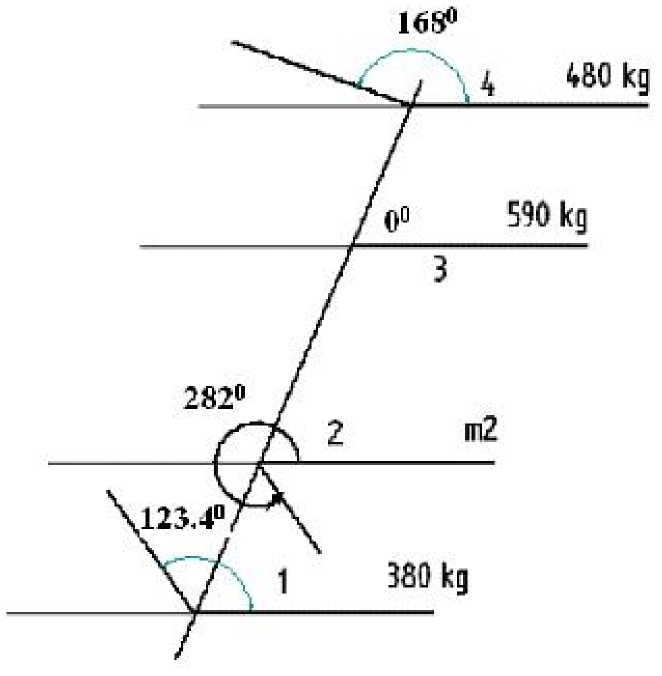
$$\mathbf{or}\ \mathbf{m}_2\ \mathbf{sin}\ \theta_2 = -417.9 \text{ --------------- } (4)$$

Squaring and adding (3) and (4), we get

$$\mathbf{m}_2 = 427.1\ \mathbf{kg\ Ans}$$

Dividing (4) by (3), we get $\tan\theta_2 = \frac{-417.9}{88.5} = -4.72$

or $\theta_2 = 282^0$ **Ans**



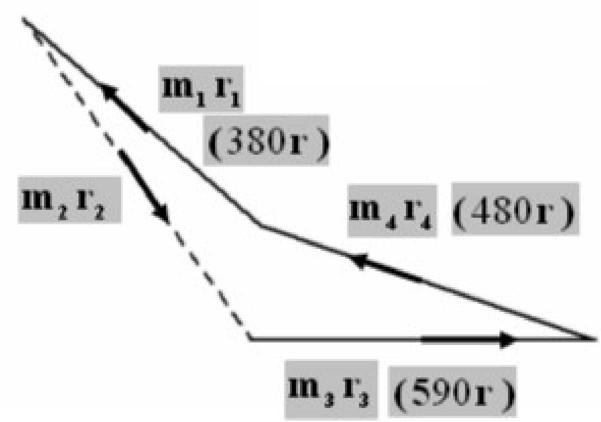
Relative crank positions

**Graphical Method:**

**Step 1:** Draw the couple diagram taking a suitable scale as shown.

This diagram provides the relative direction of the masses

**Step 2:** Now, draw the force polygon taking a suitable scale as shown.




Force diagram

**$m_1$, $m_3$ and $m_4$** .

This gives the direction and magnitude of mass $m_2$.

The results are:

26

$$_4 = 168^0 \,, \; \theta_1 = 123^0 \,, \; \theta_2 = 282^0$$

$$\mathbf{m_2\, r} = 427\mathbf{r} \;\; \textbf{or} \;\; \mathbf{m_2} = 427 \textbf{ kg Ans}$$

**Problem 2:**

Each crank of a four- cylinder vertical engine is 225 mm. The reciprocating masses of the first, second and fourth cranks are 100 kg, 120 kg and 100 kg and the planes of rotation are 600 mm, 300 mm and 300 mm from the plane of rotation of the third crank. Determine the mass of the reciprocating parts of the third cylinder and the relative angular positions of the cranks if the engine is in complete primary balance.

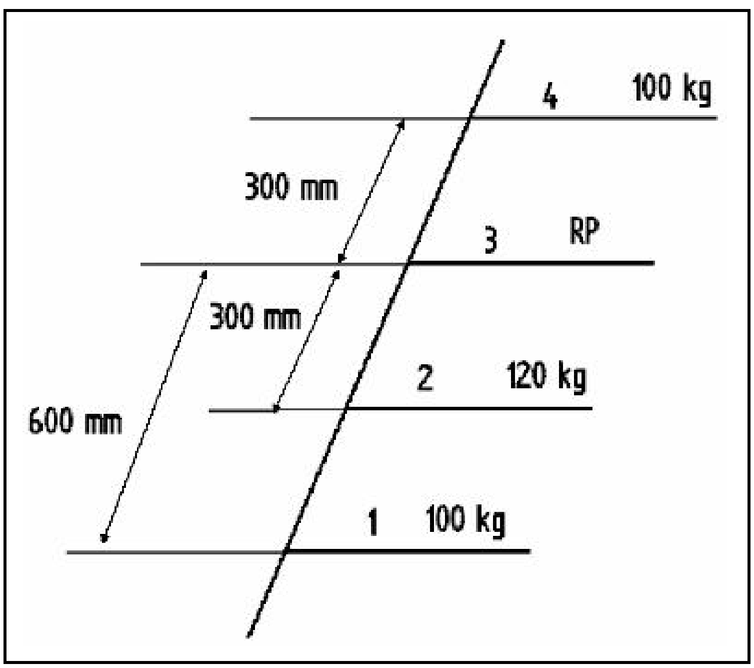
Solution:

Given :

$r = 225\ mm$

$m_1 = 100\ kg$, $m_2 = 120\ kg$ and $m_4 = 100\ kg$

| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r) kg m | Distance from Ref plane '2' m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 100 | 0.225 | 22.5 | -0.600 | -13.5 |
| 2 | 120 | 0.225 | 27.0 | -0.300 | -8.1 |
| 3(RP) | $m_3$ | 0.225 | 0.225 $m_3$ | 0 | 0 |
| 4 | 100 | 0.225 | 22.5 | 0.300 | 6.75 |



**Analytical Method:**

Choose plane 3 as the reference plane and $\theta_1 = 0^0$ .

Step 1:

Resolve the couples into their horizontal and vertical components and take their sums.
Sum of the horizontal components gives

$$-13.5 \cos 0^0 - 8.1 \cos \theta_2 + 6.75 \cos \theta_4 =$$

$$0 \text{ i.e., } -8.1 \cos \theta_2 = -6.75 \cos \theta_4 + 13.5$$

$$\text{i.e., } \quad 8.1 \cos \theta_2 = 6.75 \cos \theta_4 - 13.5 \text{———————(1)}$$

Sum of the vertical components gives

$$-13.5 \sin 0^0 - 8.1 \sin \theta_2 + 6.75 \sin \theta_4 = 0$$

$$\text{i.e., } \quad 8.1 \sin \theta_2 = 6.75 \sin \theta_4 \text{———————(2)}$$

Squaring and adding (1) and (2), we get

$$(8.1)^2 = (6.75 \cos \theta_4 - 13.5)^2 + (6.75 \sin \theta_4)^2$$

$$65.61 = 45.563 \cos^2\theta_4 - 182.25 \cos \theta_4 + 182.25 + 45.563 \sin^2\theta_4$$

$$= 45.563(\cos^2\theta_4 + \sin^2\theta_4) - 182.25 \cos \theta_4 + 182.25$$

$$= 45.563 - 182.25 \cos \theta_4 + 182.25$$

i.e., $182.25 \cos \theta_4 = 45.563 + 182.25 - 65.61 = 162.203$

Therefore, $\cos \theta_4 = \dfrac{162.203}{182.25}$ and $\theta_4 = 27.13^0$ Ans

Dividing (2) by (1), we get

$$\tan \theta_2 = \frac{6.75 \sin (27.13^0)}{6.75 \cos (27.13^0) - 13.5} = \frac{3.078}{-7.493} = -1.515$$

i.e., $\theta_2 = -22.33^0 + 180^0 = 157.67^0$

**Step 2:**

Resolve the forces into their horizontal and vertical components and take their sums.

Sum of the horizontal components gives

$22.5\cos(0^0) + 27\cos(157.67^0) + 0.225\,m_3\cos\theta_3 + 22.5\cos(27.13^0) = 0$

i.e., $22.5 - 24.975 + 0.225\,m_3\cos\theta_3 + 20.02 = 0$

i.e., $0.225\,m_3\cos\theta_3 = -17.545$ ––––––––––––– ( 3 )

And sum of the vertical components gives

$22.5\sin(0^0) + 27\sin(157.67^0) + 0.225\,m_3\sin\theta_3 + 22.5\sin(27.13^0) = 0$

i.e., $10.258 + 0.225\,m_3\sin\theta_3 + 10.26 = 0$

i.e., $0.225\,m_3\sin\theta_3 = -\,-20.518$ ––––––––––––– (4)

Squaring and adding (3) and (4), we get

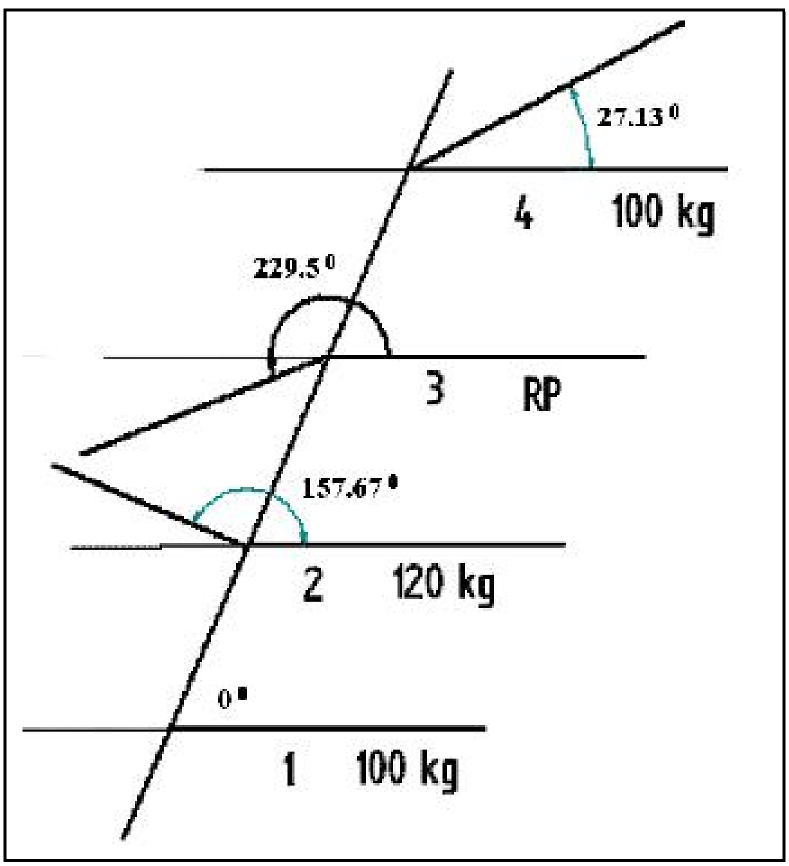
$(0.225)^2\,m_3^2 = (-17.545)^2 + (-20.518)^2$

i.e., $m_3 = \sqrt{\left(\frac{-17.545}{0.225}\right)^2 + \left(\frac{-20.518}{0.225}\right)^2}$

$= 119.98$ kg $\approx 120$ kg Ans

Dividing (4) by (3), we get $\tan\theta_3 = \frac{-20.518}{-17.545}$

or $\theta_3 = 229.5^0$ Ans



**Problem 3:**

The cranks of a four cylinder marine oil engine are at angular intervals of $90^0$. The engine speed is 70 rpm and the reciprocating mass per cylinder is 800 kg. The inner cranks are 1 m apart and are symmetrically arranged between outer cranks which are 2.6 m apart. Each crank is 400 mm long.

Determine the firing order of the cylinders for the best balance of reciprocating masses and also the magnitude of the unbalanced primary couple for that arrangement.

**Analytical Solution:**

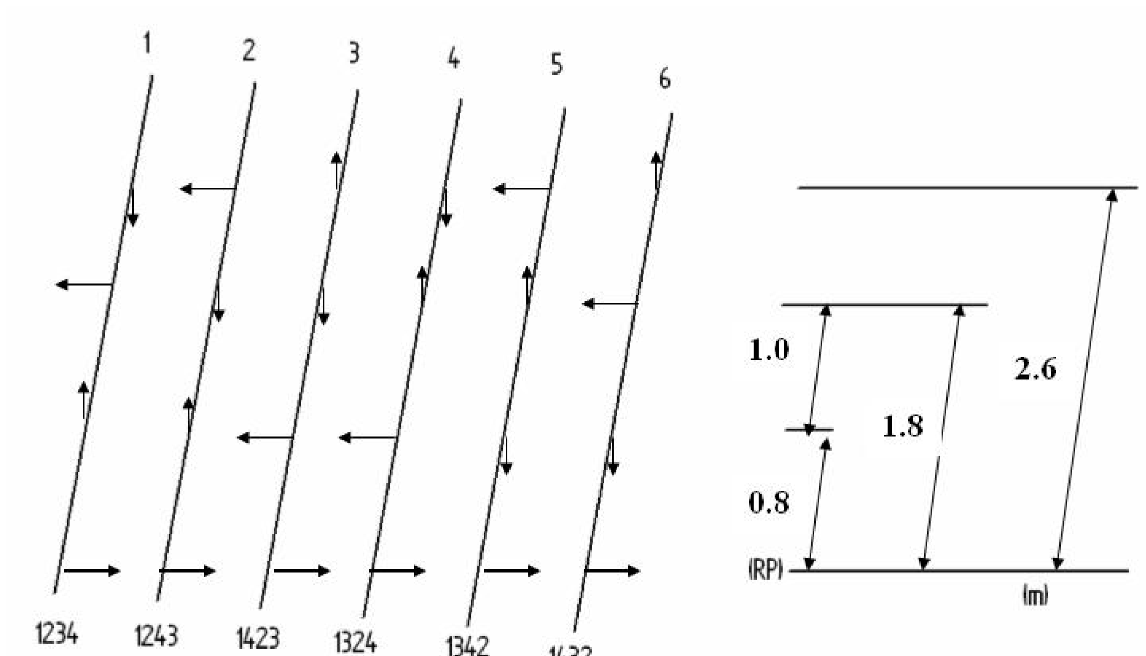
Given :

$$= = 800\ kg,\ N = 70\ rpm,\ r = 0.4\ m,\ \omega = \frac{2\pi N}{60} = 7.33\ rad/s$$

$$m r \omega^2 = 800 \times 0.4 \times (7.33)^2 = 17195$$

**Note:**

**There are four cranks. They can be used in six different arrangements as shown. It can be observed that in all the cases, primary forces are always balanced. Primary couples in each case will be as under.**




Taking 1 as the reference plane,

Firing order

Figure (a)

$$= mr\,\omega^2 \sqrt{(-l_3)^2 + (l_2 - l_4)^2} = 17195\sqrt{(-1.8)^2 + (0.8 - 2.6)^2}$$

ι 43761 **Nm**

$C_{p6} = C_{p1}$ = 43761 **Nm only, since** $l_2$ **and** $l_4$ **are int erchanged**

$$C_{p2} = mr\,\omega^2 \sqrt{(-l_4)^2 + (l_2 - l_3)^2} = 17195\sqrt{(-2.6)^2 + (0.8 - 1.8)^2}$$

= 47905 **Nm**

$C_{p5} = C_{p2}$ = 47905 **Nm only, since** $l_2$ **and** $l_3$ **are int erchanged**

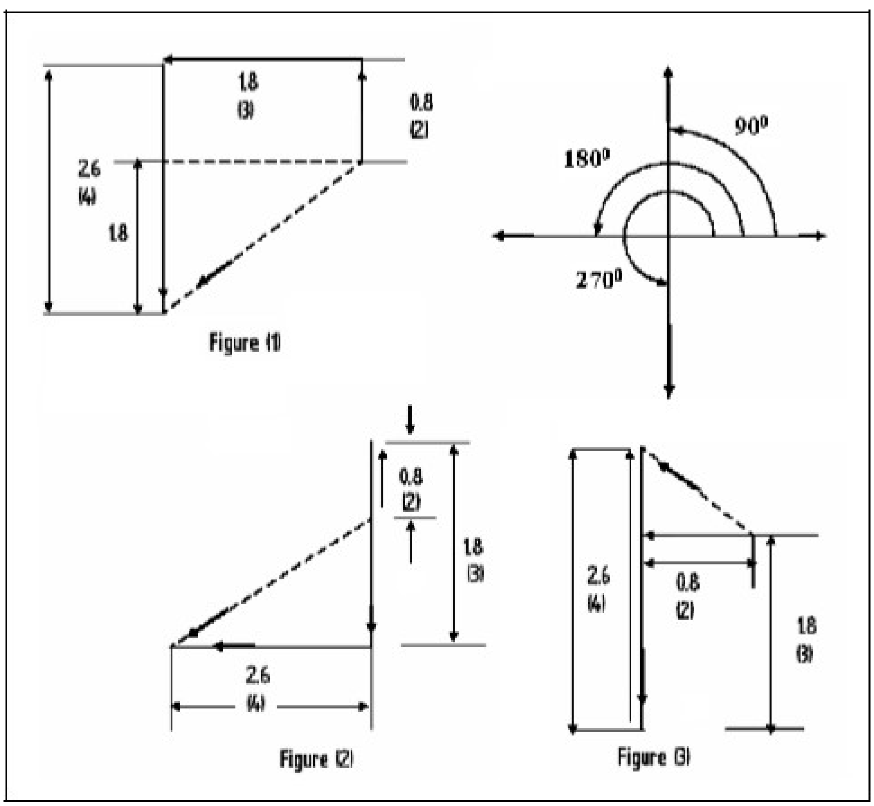
$$C_{p3} = mr\,\omega^2 \sqrt{(-l_2)^2 + (l_4 - l_3)^2} = 17195\sqrt{(-0.8)^2 + (2.6 - 1.8)^2}$$

ρ 19448 **Nm**

$C_{p4} = C_{p3}$ = 19448 **Nm only, since** $l_4$ **and** $l_3$ **are int erchanged**

**Thus the best arrangement is of 3rd and 4th. The firing orders are 1423 and 1324 respectively.**
Unbalanced couple = 19448 N m.

**Graphical solution:**



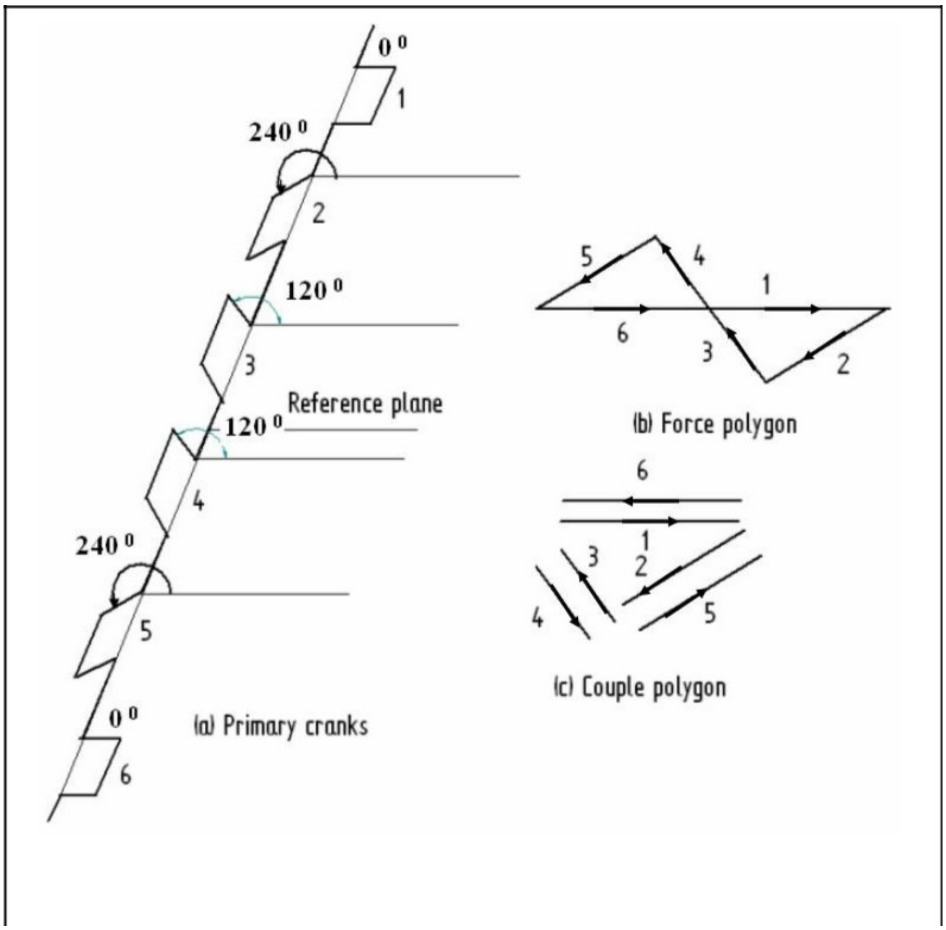
**Case 3:**

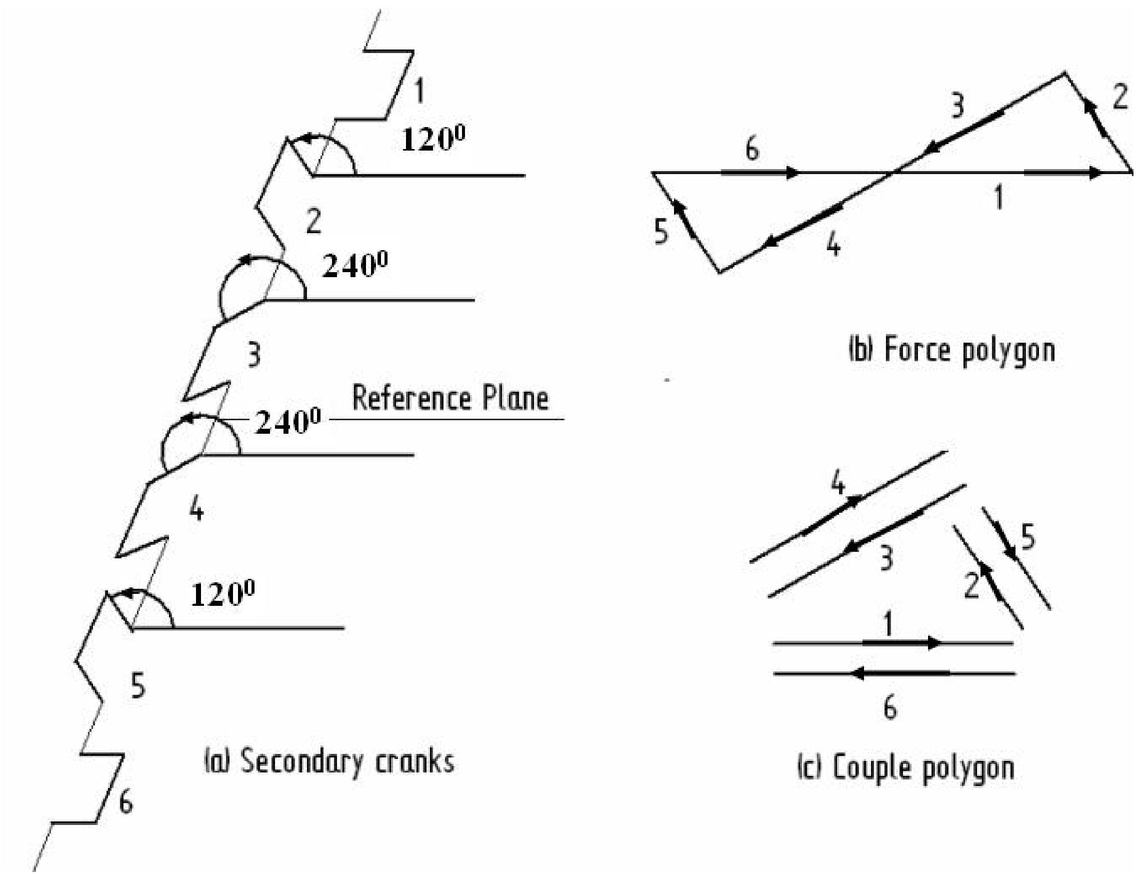
## SIX – CYLINDER, FOUR –STROKE ENGINE

Crank positions for different cylinders for the firing order 142635 for clockwise rotation of the crankshaft are, for

| | | |
|---|---|---|
| First $\theta_1 = 0^0$ | Second $\theta_2 = 240^0$ | And |
| Third $\theta_3 = 120^0$ | Fourth $\theta_4 = 120^0$ | $m_1 = m_2 = m_3 = m_4 = m_5 = m_6$ |
| Fifth $\theta_5 = 240^0$ | Sixth $\theta_6 = 0^0$ | $r_1 = r_2 = r_3 = r_4 = r_5 = r_6$ |

Since all the force and couple polygons close, it is inherently balanced engine for primary and secondary forces and couples.






**Problem 1:**

Each crank and the connecting rod of a six-cylinder four-stroke in-line engine are 60 mm and 240 mm respectively. The pitch distances between the cylinder centre lines are 80 mm, 80 mm, 100 mm, 80 mm and 80 mm respectively. The reciprocating mass of each cylinder is 1.4 kg. The engine speed is 1000 rpm. Determine the out-of-balance primary and secondary forces and couples on the engine if the firing order be 142635. Take a plane midway between the cylinders 3 and 4 as the reference plane.

Solution:

Given :

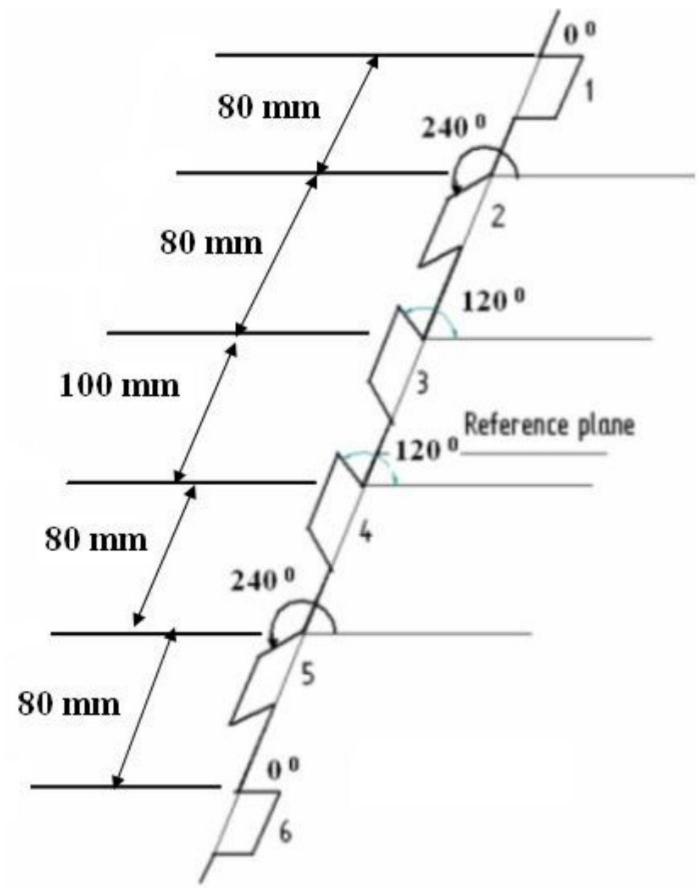
Σ = 60 mm , l = connecting rod length = 240 mm , m = reciprocating mass of each cylinder =1.4 kg , N = 1000 rpm

$$\text{We have,}\quad \omega = \frac{2\pi N}{60} = \frac{2\pi \times 1000}{60} = 104.72 \text{ rad / s}$$

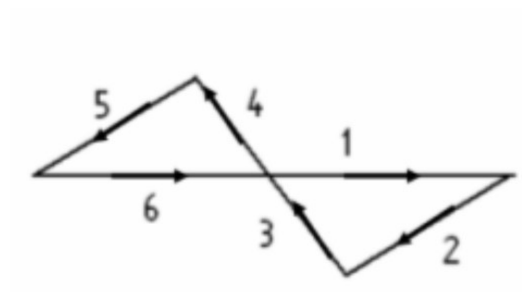
| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r ) kg m | Distance from Ref plane '2' m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 1.4 | 0.06 | 0.084 | 0.21 | 0.01764 |
| 2 | 1.4 | 0.06 | 0.084 | 0.13 | 0.01092 |
| 3 | 1.4 | 0.06 | 0.084 | 0.05 | 0.0042 |
| 4 | 1.4 | 0.06 | 0.084 | -0.05 | -0.0042 |
| 5 | 1.4 | 0.06 | 0.084 | -0.13 | -0.01092 |
| 6 | 1.4 | 0.06 | 0.084 | -0.21 | -0.01764 |

**Graphical Method:**

**Step 1:**
Draw the primary force and primary couple polygons taking some convenient scales.
**Note: For drawing these polygons take primary cranks position as the reference**

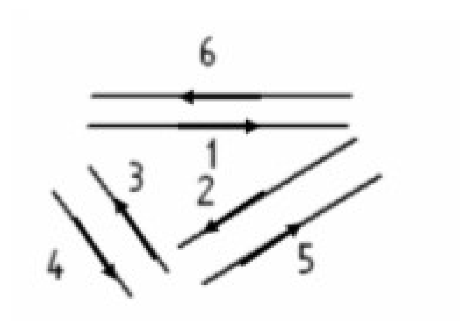


(a) Primary cranks



(b) Force polygon

**NO UNBALANCED PRIMARY FORCE**



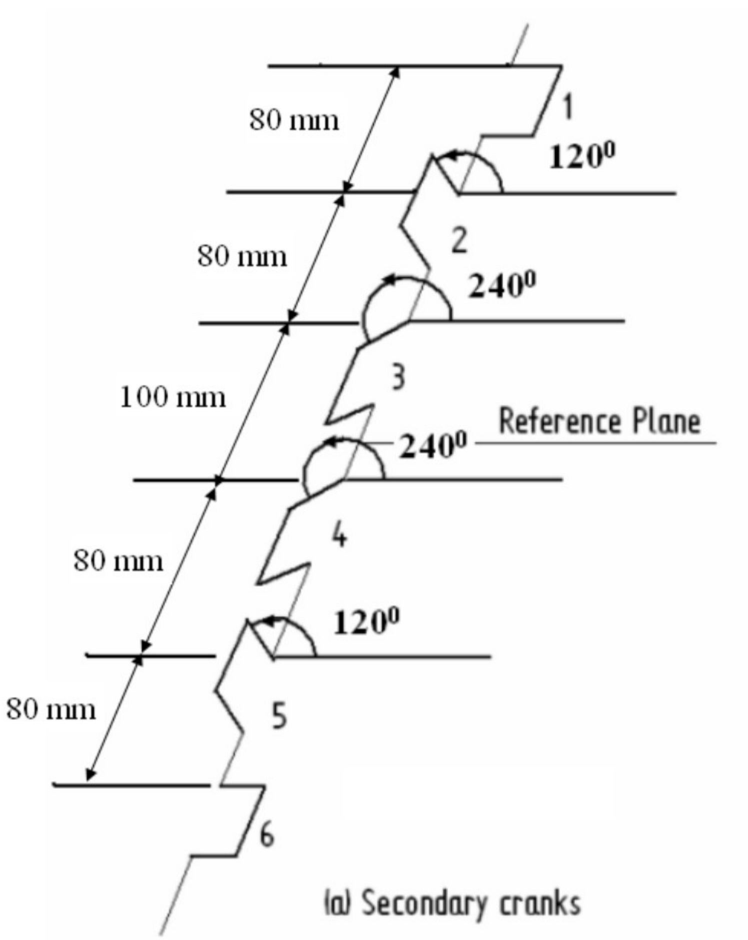
(c) Couple polygon

**NO UNBALANCED PRIMARY COUPLE**

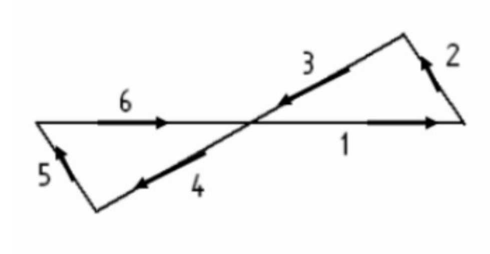
**Step 2:**

Draw the secondary force and secondary couple polygons taking some convenient scales.

**Note: For drawing these polygons take secondary cranks position as the reference**

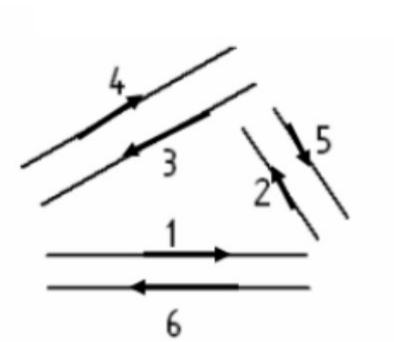



(a) Secondary cranks




(b) Force polygon

**NO UNBALANCED SECONDARY FORCE**




(c) Couple polygon

**NO UNBALANCED SECONDARY COUPLE**

**Problem 2:**

The firing order of a six –cylinder vertical four-stroke in-line engine is 142635. The piston stroke is 80 mm and length of each connecting rod is 180 mm. The pitch distances between the cylinder centre lines are 80 mm, 80 mm, 120 mm, 80 mm and 80 mm respectively. The reciprocating mass per cylinder is 1.2 kg and the engine speed is 2400 rpm. Determine the out-of-balance primary and secondary forces and couples on the engine taking a plane midway between the cylinders 3 and 4 as the reference plane.

Solution:

Given :

$r = \frac{L}{2} = \frac{80}{2} = 40$ mm , l = connecting rod length = 180 mm ,

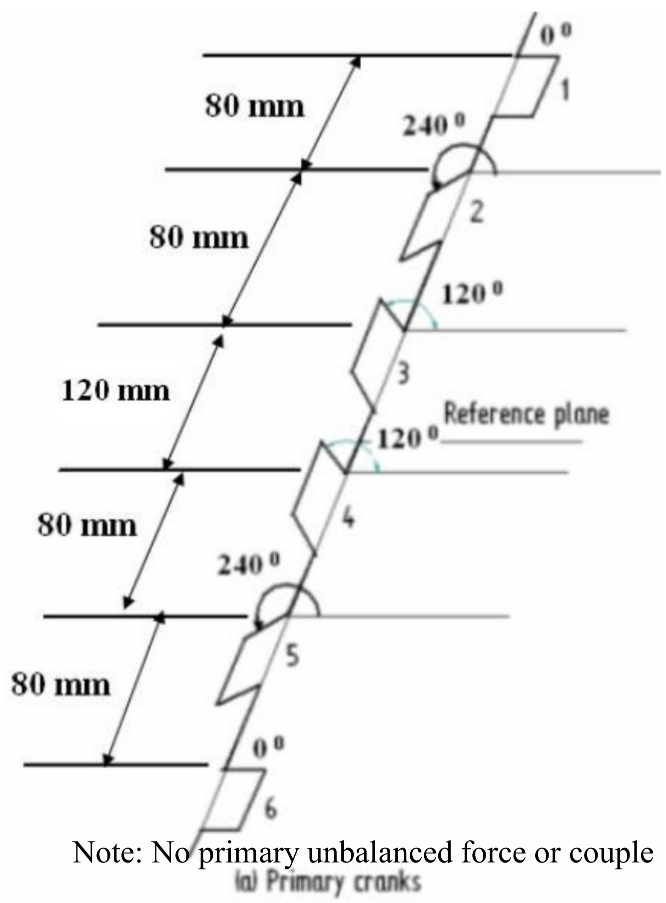
m = reciprocat ing mass of each cylinder =1.2 kg , N = 2400 rpm

We have, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 2400}{60} = 251.33$ rad / s

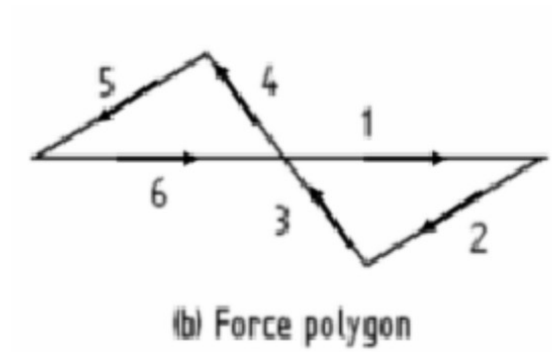
| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r ) kg m | Distance from Ref plane '2' m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 1.2 | 0.04 | 0.048 | 0.22 | 0.01056 |
| 2 | 1.2 | 0.04 | 0.048 | 0.14 | 0.00672 |
| 3 | 1.2 | 0.04 | 0.048 | 0.06 | 0.00288 |
| 4 | 1.2 | 0.04 | 0.048 | -0.06 | -0.00288 |
| 5 | 1.2 | 0.04 | 0.048 | -0.14 | -0.00672 |
| 6 | 1.2 | 0.04 | 0.048 | -0.22 | -0.01056 |

**Graphical Method:**

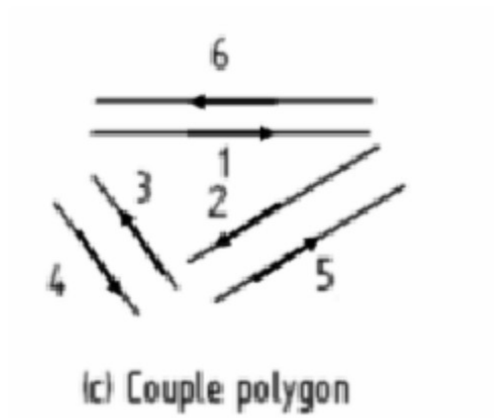
**Step 1:**
Draw the primary force and primary couple polygons taking some convenient scales.
**Note: For drawing these polygons take primary cranks position as the reference**




Note: No primary unbalanced force or couple




NO UNBALANCED PRIMARY FORCE

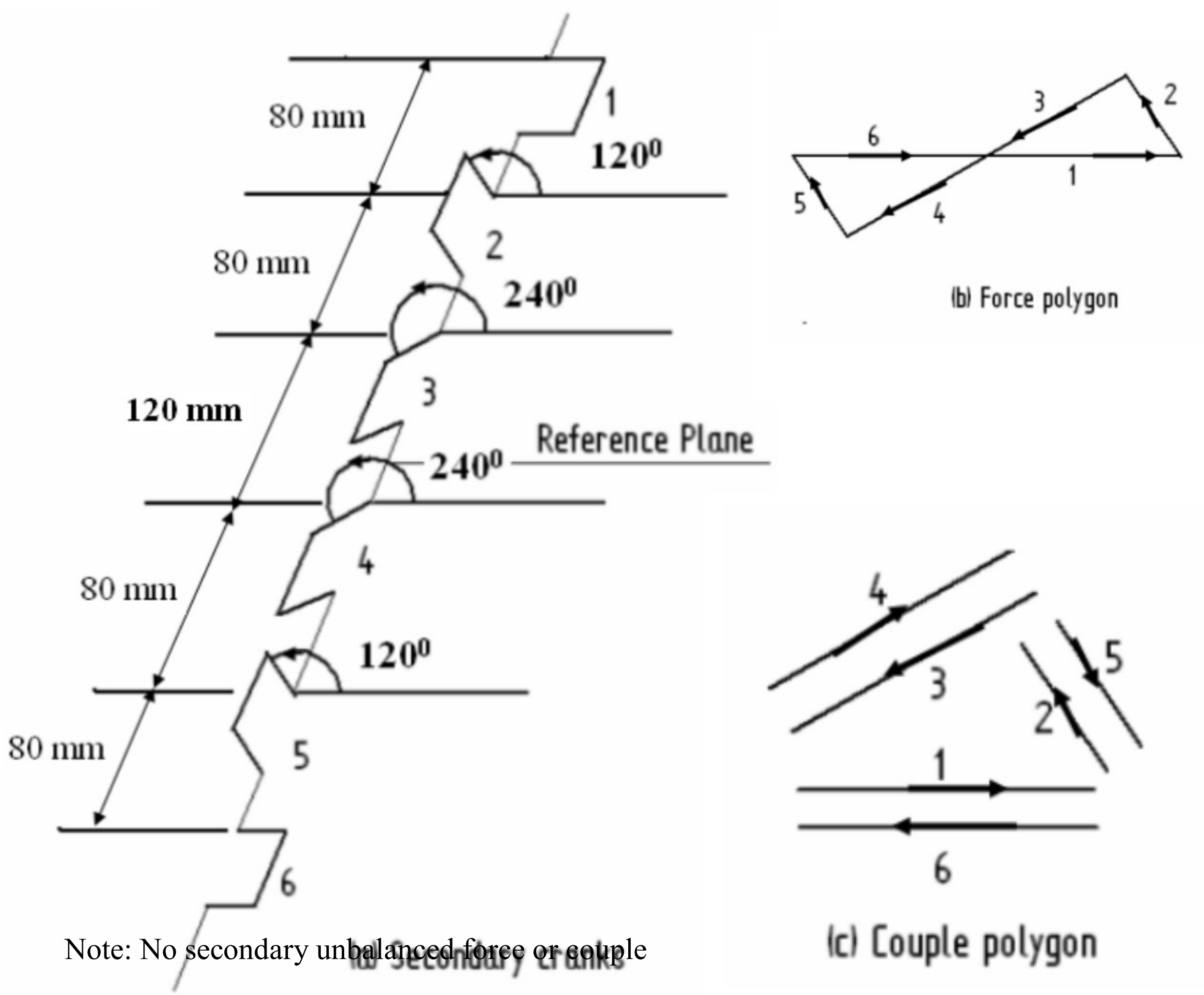



NO UNBALANCED PRIMARY COUPLE

**Step 2:**

Draw the secondary force and secondary couple polygons taking some convenient scales.

**Note: For drawing these polygons take secondary cranks position as the reference**




Note: No secondary unbalanced force or couple

**Problem 3:**

**The stroke of each piston of a six-cylinder two-stroke inline engine is 320 mm and the connecting rod is 800 mm long. The cylinder centre lines are spread at 500 mm. The cranks are at $60^0$ apart and the firing order is 145236. The reciprocating mass per cylinder is 100 kg and the rotating parts are 50 kg per crank. Determine the out of balance forces and couples about the mid plane if the engine rotates at 200 rpm.**

**Primary cranks position**

| | Relative positions of Cranks in degrees | | | | | |
|---|---|---|---|---|---|---|
| **Firing order** | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
| **142635** | **0** | **240** | **120** | **120** | **240** | **0** |
| **145236** | **0** | **180** | **240** | **60** | **120** | **300** |

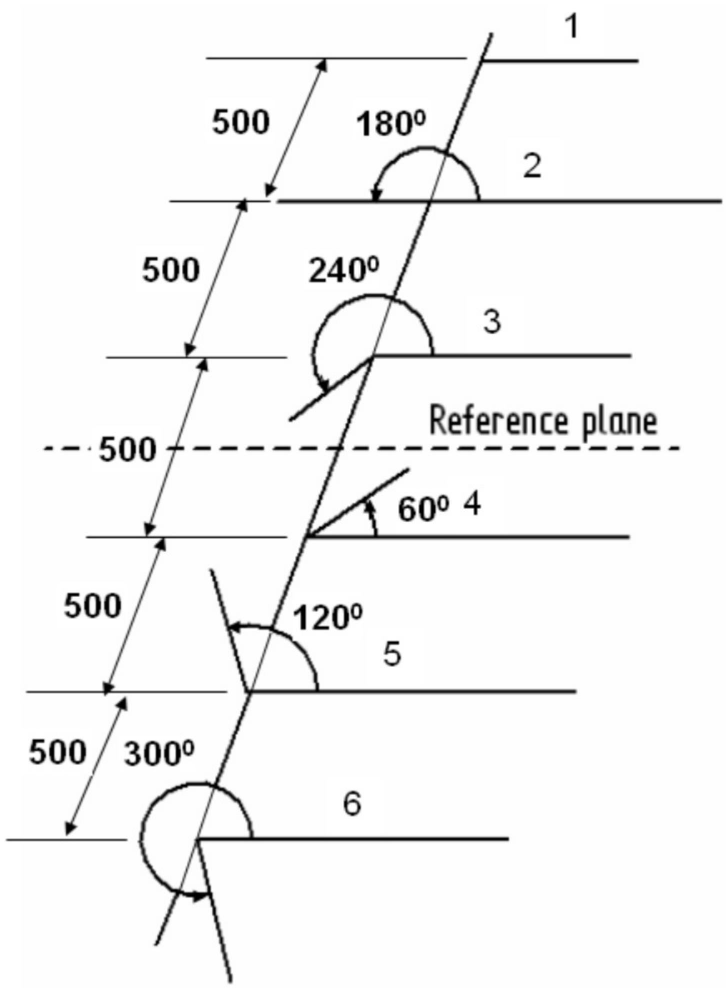
**Secondary cranks position**

| | Relative positions of Cranks in degrees | | | | | |
|---|---|---|---|---|---|---|
| **Firing order** | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
| **142635** | **0** | **120** | **240** | **240** | **120** | **0** |
| **145236** | **0** | **0** | **120** | **120** | **240** | **240** |

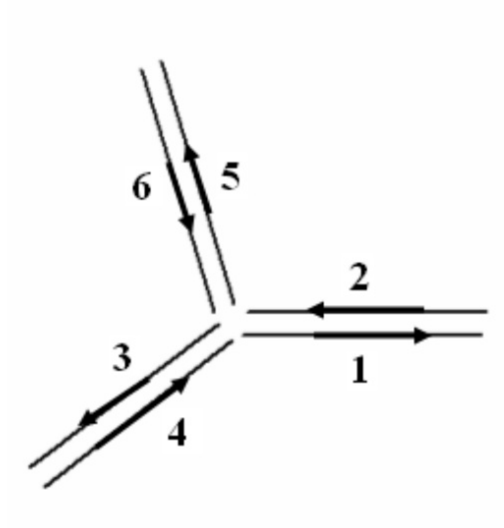
**Calculation of primary forces and couples:**

**Total mass at the crank pin = 100 kg + 50 kg = 150 kg**

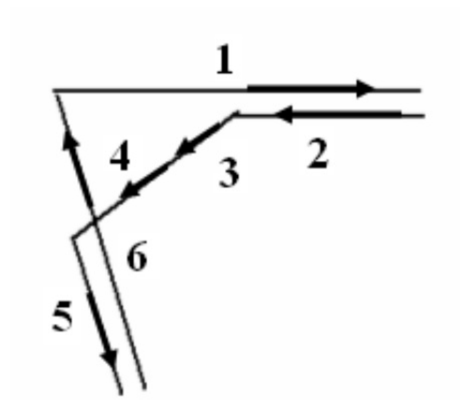
| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r ) kg m | Distance from Ref plane m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 150 | 0.16 | 24 | 1.25 | 30 |
| 2 | 150 | 0.16 | 24 | 0.75 | 18 |
| 3 | 150 | 0.16 | 24 | 0.25 | 6 |
| 4 | 150 | 0.16 | 24 | -0.25 | -6 |
| 5 | 150 | 0.16 | 24 | -0.75 | -18 |
| 6 | 150 | 0.16 | 24 | -1.25 | -30 |



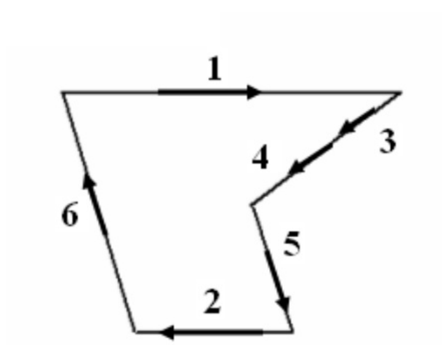

(a) Primary cranks




(b) Force polygon




(c) Couple polygon

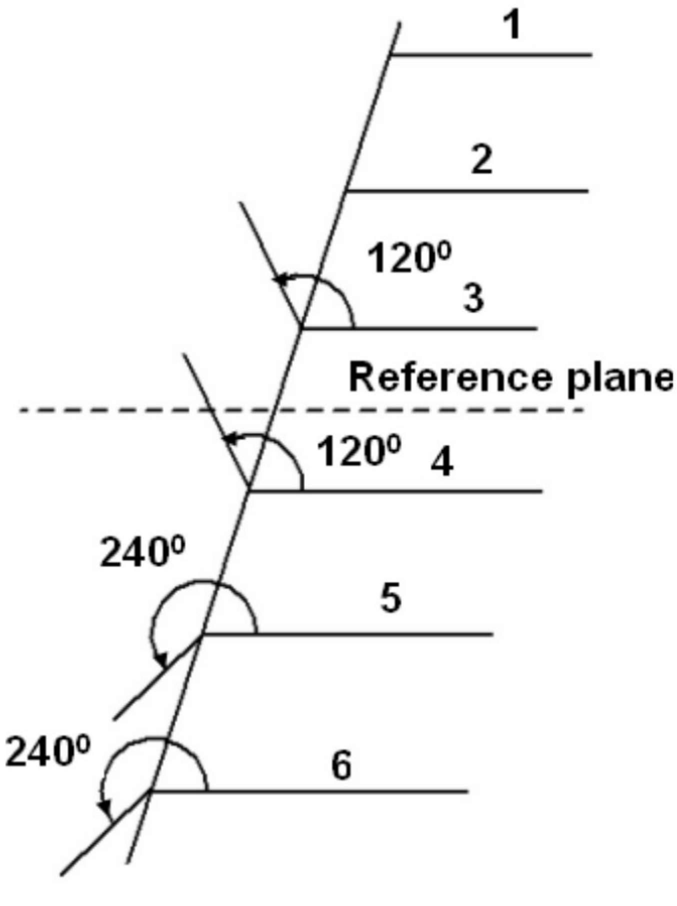



(d) Couple polygon

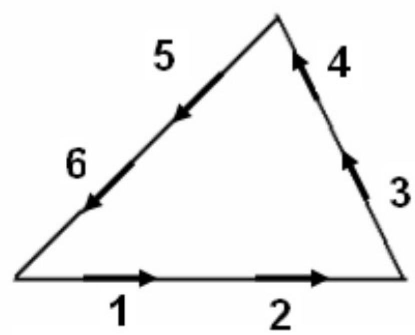
**Calculation of secondary forces and couples:**

**Since rotating mass does not affect the secondary forces as they are only due to second harmonics of the piston acceleration, the total mass at the crank is taken as 100 kg.**

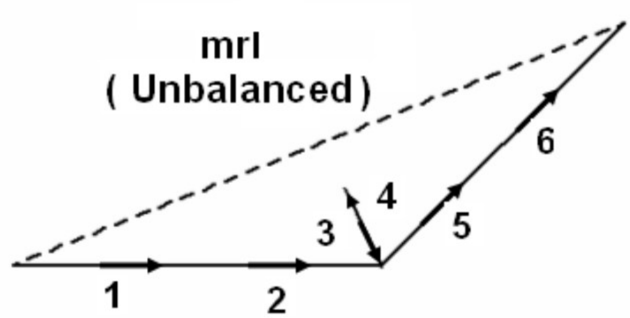
| Plane | Mass (m) kg | Radius (r) m | Cent. Force/$\omega^2$ (m r) kg m | Distance from Ref plane m | Couple/ $\omega^2$ ( m r l ) kg $m^2$ |
|---|---|---|---|---|---|
| 1 | 100 | 0.16 | 16 | 1.25 | 20 |
| 2 | 100 | 0.16 | 16 | 0.75 | 12 |
| 3 | 100 | 0.16 | 16 | 0.25 | 4 |
| 4 | 100 | 0.16 | 16 | -0.25 | -4 |
| 5 | 100 | 0.16 | 16 | -0.75 | -12 |
| 6 | 100 | 0.16 | 16 | -1.25 | -20 |




(e) Secondary cranks



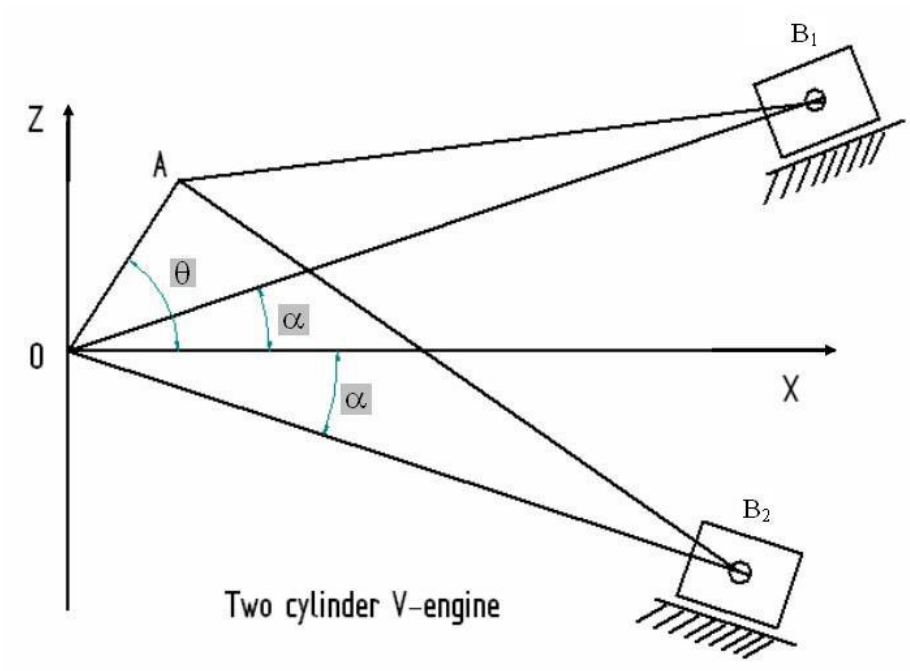

(f) Force polygon




(g) Couple polygon

# BALANCING OF V – ENGINE

**Two Cylinder V-engine:**




A common crank OA is operated by two connecting rods. The centre lines of the two – cylinders are inclined at an angle α to the X-axis.
Let θ be the angle moved by the crank from the X-axis.

**Determination of Primary force:**

Primary force of 1 along line of stroke $OB_1$ = $\mathbf{mr\,\omega^2 \cos(\theta-\alpha)}$ – – – – – –(1)

Primary force of 1 along X - axis = $\mathbf{mr\,\omega^2 \cos(\theta-\alpha)\cos\alpha}$ – – – (2)

Primary force of 2 along line of stroke $OB_2$ = $\mathbf{mr\,\omega^2 \cos(\theta+\alpha)}$ – – – – – (3)

Primary force of 2 along X-axis = $\mathbf{mr\,\omega^2 \cos(\theta+\alpha)\cos\alpha}$ – – – (4)

Total primary force al ong X - axi s

$= mr\,\omega^2 \cos\alpha\,[\cos(\theta-\alpha)+\cos(\theta+\alpha)]$

$= mr\,\omega^2 \cos\alpha\,[\cos\theta\cos\alpha+\sin\theta\sin\alpha+\cos\theta\cos\alpha-\sin\theta\sin\alpha]$

$= mr\,\omega^2 \cos\alpha \times 2\cos\theta\cos\alpha$

$= 2\,mr\,\omega^2 \cos^2\alpha\cos\theta$ – – – – – – – – – – – –(5)

Similarly,

Total primary force along Z - axis

$$+ mr\omega^2 [\cos(\theta - \alpha) \sin\alpha - \cos(\theta + \alpha) \sin\alpha]$$

$$+ mr\omega^2 \sin\alpha [(\cos\theta \cos\alpha + \sin\theta \sin\alpha) - (\cos\theta \cos\alpha - \sin\theta \sin\alpha]$$

$$+ mr\omega^2 \sin\alpha \times 2 \sin\theta \sin\alpha$$

$$+ 2 mr\omega^2 \sin^2\alpha \sin\theta \text{ ------------(6)}$$

Resultant Primary force

$$= \sqrt{(2 m r \omega^2 \cos^2\alpha \cos\theta)^2 + (2 m r \omega^2 \sin^2\alpha \sin\theta)^2}$$

$$= 2 m r \omega^2 \sqrt{(\cos^2\alpha \cos\theta)^2 + (\sin^2\alpha \sin\theta)^2} \text{ ----- (7)}$$

and this resultant primary force will be at angle β with the X – axis, given by,

$$\tan\beta = \frac{\sin^2\alpha \sin\theta}{\cos^2\alpha \cos\theta} \text{ ------(8)}$$

If $2\alpha = 90^0$, the resultant force will be equal to

$$2 m r \omega^2 \sqrt{(\cos^2 45^0 \cos\theta)^2 + (\sin^2 45^0 \sin\theta)^2}$$

$$= m r \omega^2 \text{ ----- (9)}$$

and

$$\tan\beta = \frac{\sin^2 45^0 \sin\theta}{\cos^2 45^0 \cos\theta} = \tan\theta \text{ ------(10)}$$

i.e., β = θ or it acts along the crank and therefore, can be completely balanced by a mass at a suitable radius diametrically opposite to the crank, such that,

$$m_r r_r = mr \text{ - - - - -(11)}$$

For a given value of α, the resultant force is maximum (Primary force), when

$$(\cos^2\alpha \cos\theta)^2 + (\sin^2\alpha \sin\theta)^2 \text{ is maximum}$$

or

$$(\cos^4\alpha \cos^2\theta + \sin^4\alpha \sin^2\theta) \text{ is maximum}$$

Or

$$\frac{d}{d\theta}\left(\cos^4\alpha \cos^2\theta + \sin^4\alpha \sin^2\theta\right) = 0$$ i.e., $-\cos^4\alpha \times$

$2\cos\theta \sin\theta + \sin^4\alpha \times 2\sin\theta\cos\theta = 0$

i.e., $-\cos^4\alpha \times \sin 2\theta + \sin^4\alpha \times \sin 2\theta =$
$0$ i.e., $\sin 2\theta\left[\sin^4\alpha - \cos^4\alpha\right] = 0$

As $\alpha$ is not zero, therefore for a given value of $\alpha$, the resultant primary force is maximum when $\theta = 0^0$.

**Determination of Secondary force:**

Secondary force of 1 along line of stroke $OB_1$ is equal to

$$\frac{mr\omega^2}{n}\cos 2(\theta-\alpha) ------(1)$$

Secondary force of 1 along X - axis $= \frac{mr\omega^2}{n}\cos 2(\theta-\alpha)\cos\alpha --- (2)$

Secondary force of 2 along line of stroke $OB_2$ =

$$\frac{mr\omega^2}{n}\cos 2(\theta+\alpha) ----- (3)$$

Primary force of 2 along X-axis $= \frac{mr\omega^2}{n}\cos 2(\theta+\alpha)\cos\alpha --- (4)$

Therefore,

Total secondary force along X - axis

$$= \frac{mr\omega^2}{n}\cos\alpha\left[\cos 2(\theta-\alpha) + \cos 2(\theta+\alpha)\right]$$

$$= \frac{mr\omega^2}{n}\cos\alpha\left[(\cos 2\theta\cos 2\alpha + \sin 2\theta\sin 2\alpha) + (\cos 2\theta\cos 2\alpha - \sin 2\theta\sin 2\alpha)\right]$$

$$= \frac{2mr\omega^2}{n}\cos\alpha\ \cos 2\theta\cos 2\alpha ------------(5)$$

Similarly,

Total secondary force along Z - axis

$$+\frac{2\,mr\,\omega^2}{n}\sin\alpha\,\sin 2\theta\,\sin 2\alpha \quad ----------(6)$$

Resultant Secondary force

$$= \frac{2\,mr\,\omega^2}{n}\sqrt{(\cos\alpha\cos 2\theta\cos 2\alpha)^2 + (\sin\alpha\sin 2\theta\sin 2\alpha)^2} \quad -----(7)$$

And

$$\tan\beta' = \frac{\sin\alpha\sin 2\theta\sin 2\alpha}{\cos\alpha\cos 2\theta\cos 2\alpha} \quad ------(8)$$

If $2\alpha = 90^0$ or $\alpha = 45^0$,

$$\text{Secondary force} = \frac{2mr\,\omega^2}{n}\sqrt{\left(\frac{\sin 2\theta}{\sqrt{2}}\right)^2} = \sqrt{2}\,\frac{mr\,\omega^2}{n}\sin 2\theta \quad ----(9)$$

And $\tan\beta' = \infty$ and $\beta' = 90^0$ ------(10) i.e., the force acts along Z-axis and is a harmonic force and special methods are needed to balance it.

**Problem 1:**

The cylinders of a twin V-engine are set at $60^0$ angle with both pistons connected to a single crank through their respective connecting rods. Each connecting rod is 600 mm long and the crank radius is 120 mm. The total rotating mass is equivalent to 2 kg at the crank radius and the reciprocating mass is 1.2 kg per piston. A balance mass is also fitted opposite to the crank equivalent to 2.2 kg at a radius of 150 mm. Determine the maximum and minimum values of the primary and secondary forces due to inertia of the reciprocating and the rotating masses if the engine speed is 800 mm.

**Solution:**

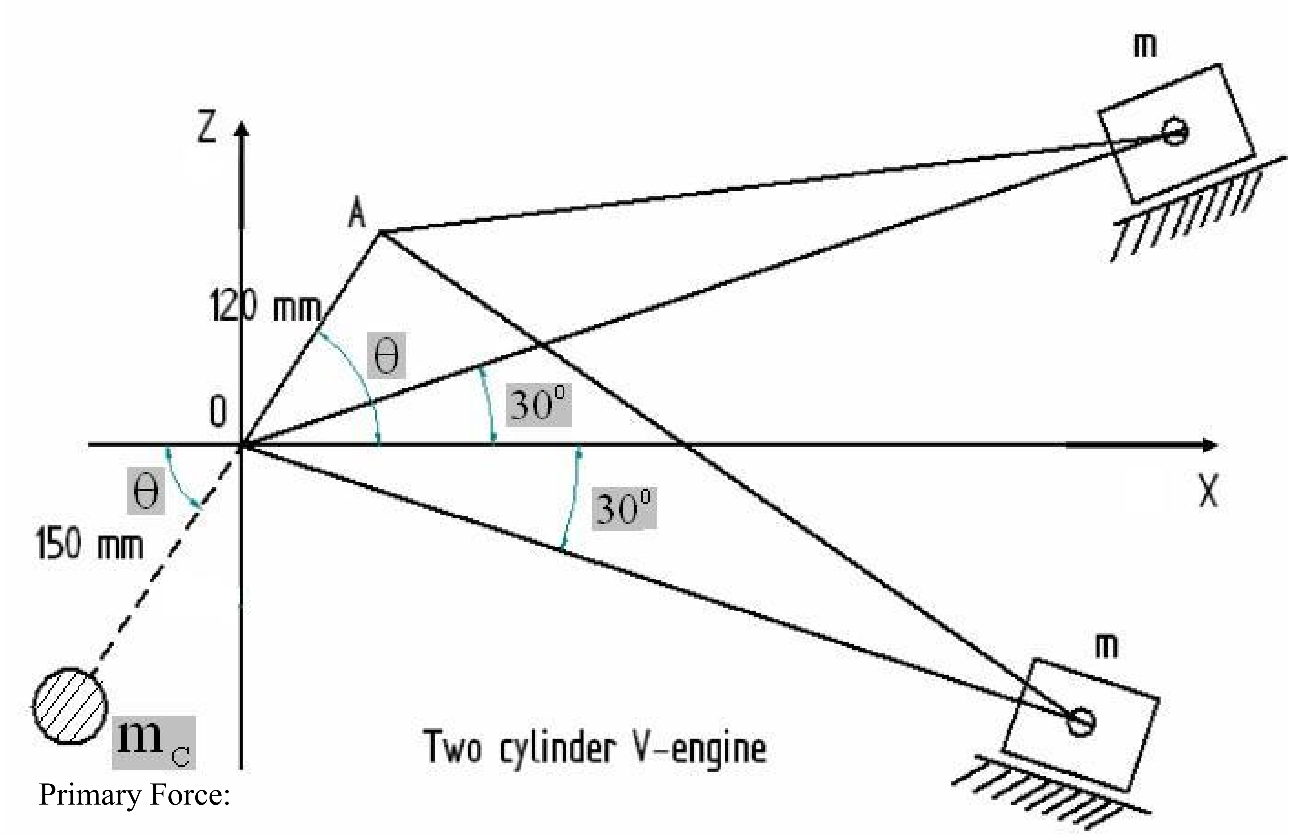
Given :

m = reciprocating mass of each piston =1.2 kg M = equivalent rotating mass = 2 kg

$m_C$ =balancing mass = 2.2 kg, $r_C$ = 150 mm l = connecting rod length = 600 mm

+ = crank radius =120 mm N = 800 rpm

We have, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 800}{60} = 83.78$ rad / s and $n = \frac{l}{r} = \frac{600}{120} = 5$




Primary Force:

Total primary force along X - axis $= 2\,m\,r\,\omega^2 \cos^2 \alpha \cos\theta$ (1) Centrifuga l force due to rotating mass along X – axis

$= M\,r\,\omega^2 \cos\theta$ ––––––––––––(2)

Centrifuga l force due to balancing mass along X – axis

$= -\,m_C\,r_C\,\omega^2 \cos\theta$ ––––––––––––(3)

Therefore total unbalance force along X –axis = (1) + (2) + (3)

That is

Total Unbalance force along X axis

$= 2\,mr\,\omega^2 \cos^2 \alpha \cos\theta + M\,r\,\omega^2 \cos\theta - m_C r_C \omega^2 \cos\theta$

$= \omega^2 \cos\theta\,[2\,mr \cos^2\alpha + M\,r - m_C\,r_C]$

$= (83.78)^2 \cos\theta\,[2x1.2x0.12x\cos^2 30^0 + 2x0.12 - 2.2x0.15]$

$= (83.78)^2 \cos\theta[0.216 + 0.24 - 0.33] = 884.41 \cos\theta$ N - - - - - - - (4)

Total primary force along Z - axis $= 2\,m\,r\,\omega^2 \sin^2\alpha \sin\theta$ ––––––––––––(5)

Centrifugal force due to rotating mass along Z – axis

$$= M\, r\, \omega^2 \sin\theta \text{ ------------(6)}$$

Centrifugal force due to balancing mass along Z – axis

$$= -m_c r_c \omega^2 \sin\theta \text{ ------------(7)}$$

Therefore total unbalance force along Z –axis = (5) + (6) + (7)

That is

Total Unbalance force along Z - axis

$$= 2\, mr\, \omega^2 \sin^2 \alpha \sin\theta + M\, r\, \omega^2 \sin\theta - m_C r_C \omega^2 \sin\theta$$

$$= \omega^2 \sin\theta \left[2\, mr \sin^2 \alpha + M\, r - m_C\, r_C\right]$$

$$= (83.78)^2 \sin\theta \left[2 x 1.2 x 0.12 x \sin^2 30^0 + 2 x 0.12 - 2.2 x 0.15\right]$$

$$= (83.78)^2 \sin\theta [0.072 + 0.24 - 0.33] = -126.34 \sin\theta \text{ N - - - - - - - (8)}$$

Resultant Primary force

$$= \sqrt{(884.41 \cos\theta)^2 + (-126.34 \sin\theta)^2}$$

$$= \sqrt{782181.05 \cos^2\theta + 15961.8 \sin^2\theta}$$

$$= \sqrt{766219.25 \cos^2\theta + 15961.8} \text{ ----- (9)}$$

This is maximum, when $\theta = 0^0$ and minimum, when $\theta = 90^0$

Maximum Primary force, i.e., when $\theta = 0^0$

$$= \sqrt{766219.25 + 15961.8} = 884.41 \text{ N -----(10)}$$

And Minimum Primary force, i.e., when $\theta = 90^0$

$$= \sqrt{766219.25 \cos^2 90^0 + 15961.8} = 126.34 \text{ N ----- (11)}$$

**Secondary force:**

The rotating masses do not affect the secondary forces as they are only due to second harmonics of the piston acceleration.

Resultant Secondary force

$$= \frac{2 mr \omega^2}{n} \sqrt{(\cos \alpha \cos 2\theta \cos 2\alpha)^2 + (\sin \alpha \sin 2\theta \sin 2\alpha)^2}$$

$$= \frac{2 \times 1.2 \times 0.12 \times (83.78)^2}{5} \sqrt{(\cos 30^2 \ \cos 2\theta \cos 60^2)^2 + (\sin 30^0 \ \sin 2\theta \sin 60^0)^2}$$

$$404.3 \sqrt{[0.1875 (\cos 2\theta)^2 + 0.1875 (\sin 2\theta)^2]} -----(12)$$

This is maximum, when $\theta = 0^0$ and minimum, when $\theta = 180^0$

Maximum secondary force, i.e., when $\theta = 0^0$

$$= 404.3 \sqrt{[0.1875 (\cos 0^0)^2 + 0.1875 (\sin 0^0)^2]} = 175.07 \text{ N} ----- (13)$$

And Minimum secondary force, i.e., when $\theta = 180^0$

$$+ 404.3 \sqrt{[0.1875 (\cos 180^0)^2 + 0.1875 (\sin 180^0)^2]} = 175.07 \text{ N} ----- (14)$$

## BALANCING OF W, V-8 AND V-12 – ENGINES

### BALANCING OF W ENGINE

In this engine three connecting rods are operated by a common crank.

Total primary force along X - axis

$$= m r \omega^2 \cos \theta (2 \cos^2 \alpha + 1) ------------(1)$$

Total primary force along Z - axis will be same a s in the V – twin engine, (since the primary force of 3 along Z – axis is zero)

$$= 2 m r \omega^2 \sin^2 \alpha \sin \theta ------------(2)$$

Resultant P r i mar y force

$$= m r \omega^2 \sqrt{[\cos \theta (2 \cos^2 \alpha + 1)^2 + (2 \sin^2 \alpha \sin \theta)^2]} ----- (3)$$

and this resultant primary force will be at angle β with the X – axis, given by,

$$\tan \beta = \frac{2\sin^2 \alpha \sin \theta}{\cos \theta(2\cos^2 \alpha + 1)} \quad \text{------}(4)$$

If $\alpha = 60^0$,

Resultant Primary force

$$= \frac{3}{2} m r \omega^2 \quad \text{-----} (5)$$

and

$$\tan \beta = = \tan \theta \quad \text{------}(6)$$

i.e., $\beta = \theta$ or it acts along the crank and therefore, can be completely balanced by a mass at a suitable radius diametrically opposite to the crank, such that, $m_r r_r = mr$ - - - - - (7 )

Total secondary force along X - axis

$$= \cos 2\theta \frac{2m r \omega^2}{n} \cos \alpha \cos 2\alpha + 1 \quad \text{-----------}(8)$$

Total secondary force along Z –direction will be same as in the V-twin engine.

Resultant secondary force

$$= \frac{m r \omega^2}{n} \sqrt{\left[\left(\cos 2\theta (2\cos \alpha \cos 2\alpha + 1)\right)^2 + \left(2\sin \alpha \sin 2\alpha \sin 2\theta\right)^2\right]} \quad \text{-----}(9)$$

$$\tan \beta' = \frac{2\sin \alpha \sin 2\theta \sin 2\alpha}{\cos 2\theta(2\cos \alpha \cos 2\alpha + 1)} \quad \text{------}(10)$$

If $\alpha = 60^0$,

Secondary force along X - axis

$$= \frac{m r \omega^2}{2n} \cos 2\theta \quad \text{-------------}(11)$$

Secondary force along Z - axis

$$\frac{3m r \omega^2}{2n} \sin 2\theta \quad \text{-------------}(12)$$

It is not possible to balance these forces simultaneously

**V-8 ENGINE**

It consists of two banks of four cylinders each. The two banks are inclined to each other in the shape of V. The analysis will depend on the arrangement of cylinders in each bank.

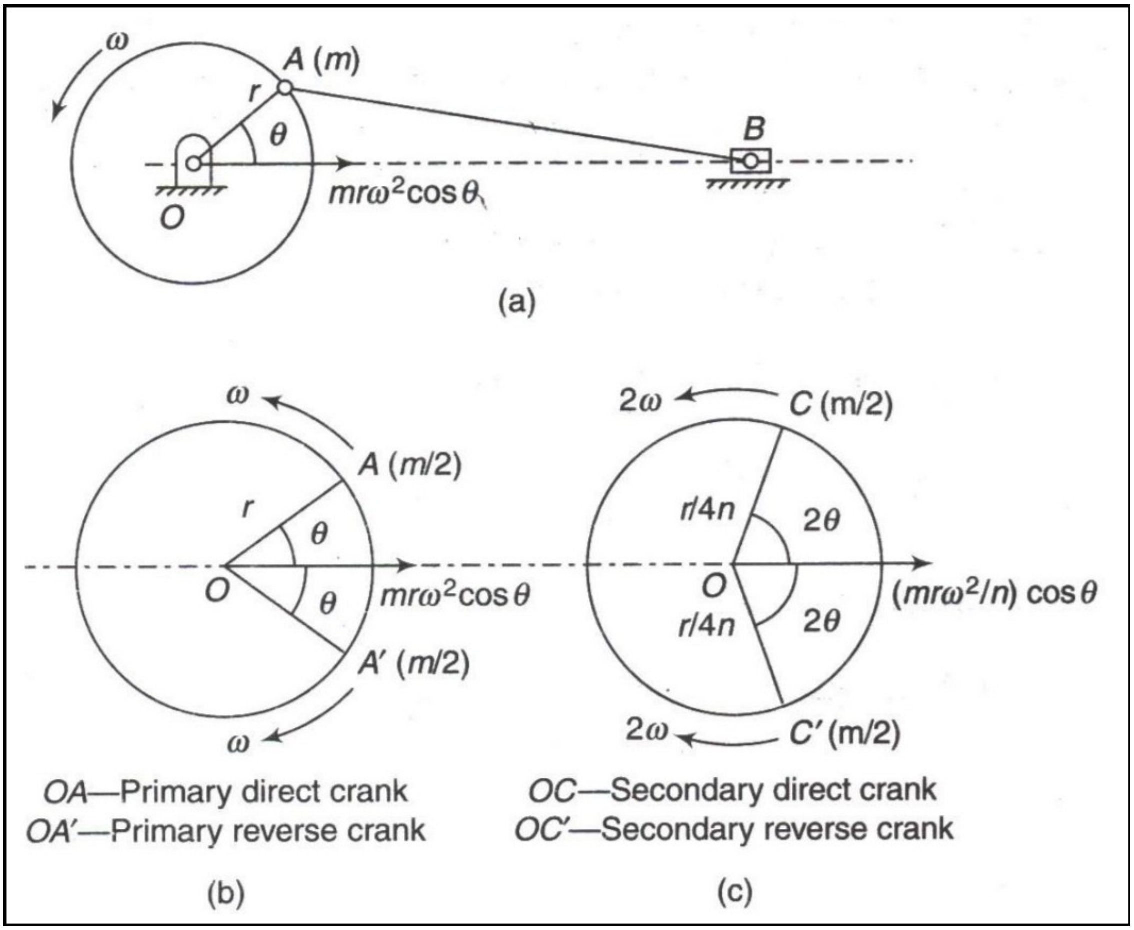
**V-12 ENGINE**

It consists of two banks of six cylinders each. The two banks are inclined to each other in the shape of V. The analysis will depend on the arrangement of cylinders in each bank.

If the cranks of the six cylinders on one bank are arranged like the completely balanced six cylinder, four stroke engine then, there is no unbalanced force or couple and thus the engine is completely balanced.

**BALANCING OF RADIAL ENGINES:**

It is a multicylinder engine in which all the connecting rods are connected to a common crank.



**Direct and reverse crank method of analysis:**

In this all the forces exists in the same plane and hence no couple exist.

In a reciprocating engine the primary force is given by, $\mathbf{mr}\,\omega^2\,\mathbf{cos}\theta$ which acts along the line of stroke.

In direct and reverse crank method of analysis, a force identical to this force is generated by two masses as follows.

1.A mass m/2, placed at the crank pin A and rotating at an angular velocity ω in the counter clockwise direction.
2.A mass m/2, placed at the crank pin of an imaginary crank OA' at the same angular position as the real crank but in the opposite direction of the line of stroke. It is assumed to rotate at an angular velocity ω in the clockwise direction (opposite).
3. While rotating, the two masses coincide only on the cylinder centre line.

The components of the centrifugal forces due to rotating masses along the line of stroke are,

$$\textbf{Due to mass at A} = \frac{\mathbf{m}}{2}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\,\theta$$

$$\textbf{Due to mass at A}' = \frac{\mathbf{m}}{2}\,\mathbf{r}\,\omega^2\,\mathbf{cos}\theta$$

Thus, total force along the line of stroke = $\mathbf{mr}\,\omega^2\,\mathbf{cos}\theta$ which is equal to the primary force.
At any instant, the components of the centrifugal forces of these masses normal to the line of stroke will be equal and opposite.
The crank rotating in the direction of engine rotation is known as the **direct crank** and the imaginary crank rotating in the opposite direction is known as the **reverse crank.**

**Now,**

Secondary accelerating force is

$$\mathbf{mr}\,\omega^2\,\frac{\mathbf{cos}\,2\theta}{\mathbf{n}} = \mathbf{mr}(2\omega)^2\,\frac{\mathbf{cos}\,2\theta}{4\mathbf{n}}$$

$$\mathbf{r}$$

$$\nu\nu\ \mathbf{m}\ (2\omega)^2\ \mathbf{cos}\ 2\,\theta\ 4\mathbf{n}$$

This force can also be generated by two masses in a similar way as follows.

1. A mass m/2, placed at the end of direct secondary crank of length $\frac{r}{4n}$ at an angle 2θ and rotating at an angular velocity 2ω in the counter clockwise direction.

= A mass m/2, placed at the end of reverse secondary crank of length $\frac{r}{4n}$ at an angle -2θ and rotating at an angular velocity 2ω in the clockwise direction.

The components of the centrifugal forces due to rotating masses along the line of stroke are,

$$\text{Due to mass at C} = \frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{2n}\cos 2\theta$$

$$\text{Due to mass at C'} = \frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{2n}\cos 2\theta$$

Thus, total force along the line of stroke =

$2 \times \frac{m}{2}\frac{r}{4n}(2\omega)^2\cos 2\theta = \frac{mr\omega^2}{n}\cos 2\theta$ which is equal to the secondary force.

# Longitudinal and Transverse Vibrations

Features (Main)



## Introduction

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a ***vibratory motion.*** This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilib- rium position. In this way, the vibratory motion is repeated indefinitely.

## Terms Used in Vibratory Motion

The following terms are commonly used in connection with the vibratory motions :

1. ***Period of vibration or time period.*** It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. ***Cycle.*** It is the motion completed during one time period.

3. ***Frequency.*** It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

## Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view :

**1.** ***Free or natural vibrations.*** When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.*** The frequency of the free vibrations is called ***free or natural frequency.***

**2.** ***Forced vibrations.*** When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.*** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

**Note :** When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

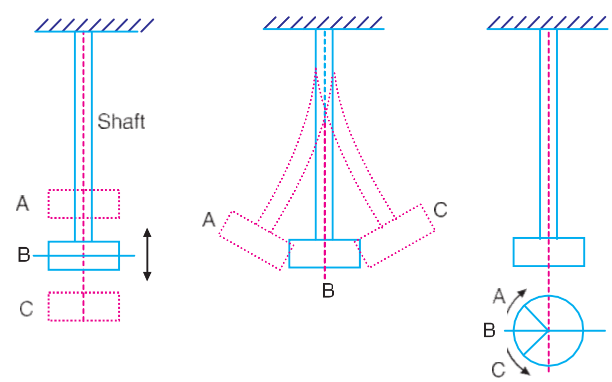
**3.** ***Damped vibrations.*** When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.*** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

## Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view :

**1.** Longitudinal vibrations, **2.** Transverse vibrations, and **3.** Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. 23.1. This system may execute one of the three above mentioned types of vibrations.




*B* = Mean position ; *A* and *C* = Extreme positions.

(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

**Fig. 23.1.** Types of free vibrations.

**1.** ***Longitudinal vibrations.*** When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 23.1 (*a*), then the vibrations are known as ***longitudinal vibrations.*** In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

**2. *Transverse vibrations.*** When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 23.1 (*b*), then the vibrations are known as ***transverse vibrations.*** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

Bridges should be built taking vibrations into account.

**3. *Torsional vibrations**.** When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. 23.1 (*c*), then the vibrations are known as ***torsional vibrations.*** In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

**Note :** If the limit of proportionality (*i.e.* stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

## Natural Fre quency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by the following three methods :

### 1. Equilibrium Method

Consider a constraint (*i.e.* spring) of negligible mass in an unstrained position, as shown in Fig. 23.2 (*a*).

Let $s$ = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

$m$ = Mass of the body suspended from the constraint in kg,

$W$ = Weight of the body in newtons = $m.g$,

* The torsional vibrations are separately discussed in chapter 24.

$\delta$ = Static deflection of the spring in metres due to weight $W$ newtons, and

$x$ = Displacement given to the body by the external force, in metres.

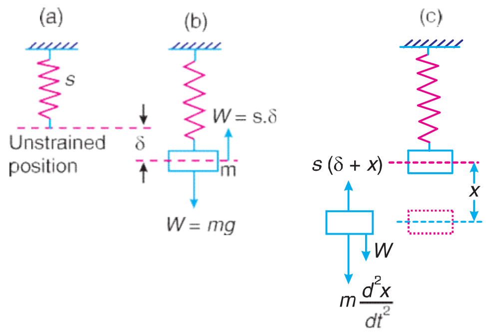



**Fig. 23.2.** Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 23.2 (*b*), the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$ .

Since the mass is now displaced from its equilibrium position by a distance $x$, as shown in Fig. 23.2 (*c*), and is then released, therefore after time $t$,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . (*i*)

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$= m \times \frac{d^2x}{dt^2}$ . . . (Taking downward force as positive) . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass $m$ after time $t$ is

$$m \times \frac{d^2x}{dt^2} = -s \cdot x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s \cdot x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \quad \text{. . . (iii)}$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{. . . (iv)}$$

Comparing equations (*iii*) and (*iv*), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because m.g = s.\delta$)

Taking the value of $g$ as 9.81 m/s$^2$ and $\delta$ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

**Note :** The value of static deflection $\delta$ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where $\delta$ = Static deflection *i.e.* extension or compression of the constraint,

$W$ = Load attached to the free end of constraint,

$l$ = Length of the constraint,

$E$ = Young's modulus for the constraint, and

$A$ = Cross-sectional area of the constraint.

## 2. Energy method

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position. In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

This industrial compressor uses compressed air to power heavy-duty construction tools. Compressors are used for jobs, such as breaking up concrete or paving, drilling, pile driving, sand-blasting and tunnelling. A compressor works on the same principle as a pump. A piston moves backwards and forwards inside a hollow cylinder, which compresses the air and forces it into a hollow chamber. A pipe or hose connected to the chamber channels the compressed air to the tools.

Note : This picture is given as additional information and is not a direct example of the current chapter.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \quad \frac{d}{dt}(K.E. + P.E.) = 0$$

We know that kinetic energy,

$$K.E. = \frac{1}{2} \times m\left(\frac{dx}{dt}\right)^2$$

and potential energy, $$P.E. = \left(\frac{0 + s.x}{2}\right)x = \frac{1}{2} \times s.x^2$$

. . . ($\because$ *P.E.* = Mean force × Displacement )

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\times s.x^2\right] = 0$$

$$\frac{1}{2}\times m \times 2 \times \frac{dx}{dt} \times \frac{d^2x}{dt^2} + \frac{1}{2} \times s \times 2x \times \frac{dx}{dt} = 0$$

or $$m \times \frac{d^2x}{dt^2} + s.x = 0 \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0$$ . . . (Same as before)

The time period and the natural frequency may be obtained as discussed in the previous method.

## 3. Rayleigh's method

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then

$$x = X \sin \omega.t \qquad \text{. . . (i)}$$

where $x$ = Displacement of the body from the mean position after time $t$ seconds, and

$X$ = Maximum displacement from mean position to extreme position.

Now, differentiating equation ***(i)***, we have

$$\frac{dx}{dt} = \omega \times X \cos \omega.t$$

Since at the mean position, $t = 0$, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega.X$$

$\therefore$ Maximum kinetic energy at mean position

$$= \frac{1}{2} \times m.v^2 = \frac{1}{2} \times m.\omega^2.X^2 \qquad \text{. . . (ii)}$$

and maximum potential energy at the extreme position

$$= \left(\frac{0 + s.X}{2}\right)X = \frac{1}{2} \times s.X^2 \qquad \text{. . . (iii)}$$

Equating equations ***(ii)*** and ***(iii)***,

$$\frac{1}{2} \times m.\omega^2.X^2 = \frac{1}{2} \times s.X^2 \quad \text{or} \quad \omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

$$\therefore \quad \text{Time period,} \quad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{s}{m}}$$ . . . (Same as before)

and natural frequency, $f_n = \dfrac{1}{t_p} = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{s}{m}}$ . . . (Same as before)

**Note :** In all the above expressions, $\omega$ is known as **natural circular frequency** and is generally denoted by $\omega_n$.

## Natural Frequency of Free Transverse Vibrations

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight $W$, as shown in Fig. 23.3.

Let $s$ = Stiffness of shaft,

$\delta$ = Static deflection due to weight of the body,

$x$ = Displacement of body from mean position after time $t$.

$m$ = Mass of body = $W/g$

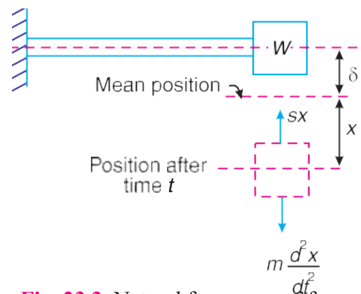



**Fig. 23.3.** Natural frequency of free transverse vibrations.

As discussed in the previous article,

Restoring force $= -\,s.x$ . . . ***(i)***

and accelerating force $= m \times \dfrac{d^2x}{dt^2}$ . . . ***(ii)***

Equating equations ***(i)*** and ***(ii)***, the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -\,s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (Same as before )}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

Time period, $t_p = 2\pi\sqrt{\dfrac{m}{s}}$

and natural frequency, $f_n = \dfrac{1}{t_p} = \dfrac{1}{2\pi}\sqrt{\dfrac{s}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta}}$

**Note :** The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where $W$ = Load at the free end, in newtons,

$l$ = Length of the shaft or beam in metres,

$E$ = Young's modulus for the material of the shaft or beam in N/m$^2$, and

$I$ = Moment of inertia of the shaft or beam in m$^4$.

**Example 23.1.** *A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m². Determine the frequency of longitudinal and transverse vibrations of the shaft.*

**Solution.** Given : $d$ = 50 mm = 0.05 m ; $l$ = 300 mm = 0.03 m ; $m$ = 100 kg ; $E$ = 200 GN/m² = 200 × 10⁹ N/m²

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3}\,\text{m}^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.3 \times 10^{-6}\,\text{m}^4$$

## Frequency of longitudinal vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.81 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^9} = 0.751 \times 10^{-6}\,\text{m}$$

...($\because W = m.g$)

∴ Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575\text{ Hz}$$ **Ans.**

## Frequency of transverse vibration

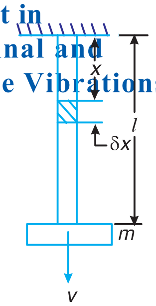
We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3\,E.I} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3}\,\text{m}$$

∴ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41\text{ Hz}$$ **Ans.**

## Effect of Inertia of the Constraint in Longitudinal and Transverse Vibrations




In deriving the expressions for natural frequency of longitudinal and transverse vibrations, we have neglected the inertia of the constraint *i.e.* shaft. We shall now discuss the effect of the inertia of the constraint, as below :

### 1. Longitudinal vibration

Consider the constraint whose one end is fixed and other end is free as shown in Fig. 23.4.

Let $m_1$ = Mass of the constraint per unit length,

$l$ = Length of the constraint,

$m_C$ = Total mass of the

constraint = $m_1 . l$, and

$v$ = Longitudinal velocity of the free end.

**Fig. 23.4.** Effect of inertia of the constraint in longitudinal vibrations.

Consider a small element of the constraint at a distance $x$ from the fixed end and of length $\delta x$.

$\therefore$ Velocity of the small element

$$= \frac{x}{l} \times v$$

and kinetic energy possessed by the element

$$= \frac{1}{2} \times \text{Mass (velocity)}^2$$

$$= \frac{1}{2} \times m_1 . \delta x \left( \frac{x}{l} \times v \right)^2 = \frac{m_1 . v^2 \, x^2}{2l^2} \times \delta x$$

$\therefore$ Total kinetic energy possessed by the constraint,

$$= \int_0^l \frac{m_1 . v^2 x^2}{2l^2} \times dx = \frac{m_1 . v^2}{2l^2} \left[ \frac{x^3}{3} \right]_0^l$$

$$= \frac{m_1 . v^2}{2l^2} \times \frac{l^3}{3} = \frac{1}{2} \times m_1 . v^2 \times \frac{l}{3} = \frac{1}{2} \left( \frac{m_1 . l}{3} \right) v^2 = \frac{1}{2} \left( \frac{m_C}{3} \right) v^2 \quad \ldots (i)$$

. . . (Substituting $m_1 . l = m_C$)

If a mass of $\frac{m_C}{3}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2} \left( \frac{m_C}{3} \right) v^2 \quad \ldots \text{[Same as equation } (i)] \ldots (ii)$$

Hence the two systems are dynamically same. Therefore, inertia of the constraint may be allowed for by adding one-third of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint $m_C$ and the mass of the disc $m$ at the end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi} \sqrt{\frac{s}{m + \frac{m_C}{3}}}$$

## 2. Transverse vibration

Consider a constraint whose one end is fixed and the other end is free as shown in Fig. 23.5.

Let $m_1$ = Mass of constraint per unit length,

$l$ = Length of the constraint,

$m_C$ = Total mass of the constraint = $m_1 . l$, and

$v$ = Transverse velocity of the free end.

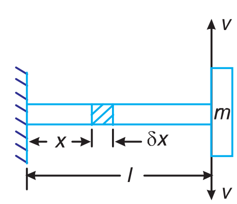


**Fig. 23.5.** Effect of inertia of the constraint in transverse vibrations.

Consider a small element of the constraint at a distance $x$ from the fixed end and of length $\delta x$. The velocity of this elementis given by $\left[\dfrac{3l.x^2 - x^3}{2l^3} \times v\right]$.

∴ Kinetic energy of the element

$$= \frac{1}{2} \times m_1.\delta x \left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2$$

and total kinetic energy of the constraint,

$$= \int_0^l \frac{1}{2} \times m_1 \left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2 dx = \frac{m_1 v^2}{8l^6} \int_0^l (9l^2.x^4 - 6l.x^5 + x^6)\, dx$$

$$= \frac{m_1.v^2}{8l^6}\left[\frac{9\, l^2.x^5}{5} - \frac{6\, l.x^6}{6} + \frac{x^7}{7}\right]_0^l$$

$$= \frac{m_1 v^2}{8 l^6}\left[\frac{9\, l^7}{5} - \frac{6 l^7}{6} + \frac{l^7}{7}\right] = \frac{m_1 v^2}{8 l^6}\left(\frac{33 l^7}{35}\right)$$

$$= \frac{33}{280} \times m_1.l.v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_1.l\right) v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_C\right) v^2 \quad \ldots (i)$$

. . . (Substituting $m_1.l = m_C$)

If a mass of $\dfrac{33\, m_C}{140}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2}\left(\frac{33\, m_C}{140}\right) v^2 \quad \ldots [\text{Same as equation } (i)]$$

Hence the two systems are dynamically same. Therefore the inertia of the constraint may be allowed for by adding $\dfrac{33}{140}$ of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint $m_C$ and the mass of the disc $m$ at the free end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \dfrac{33\, m_C}{140}}}$$

**Notes : 1.** If both the ends of the constraint are fixed, and the disc is situated in the middle of it, then proceeding in the similar way as discussed above, we may prove that the inertia of the constraint may be allowed for by adding $\dfrac{13}{35}$ of its mass to the disc.

**2.** If the constraint is like a simply supported beam, then $\dfrac{17}{35}$ of its mass may be added to the mass of the disc.

## Natural Frequency of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Consider a shaft *AB* of length *l*, carrying a point load *W* at *C* which is at a distance of $l_1$ from *A* and $l_2$ from *B*, as shown in Fig. 23.6. A little consideration will show that when the shaft is deflected and suddenly released, it will make transverse vibrations. The deflection of the shaft is proportional to the load *W* and if the beam is deflected beyond the static equilibrium position then the load will vibrate with simple harmonic motion (as by a helical spring). If δ is the static deflection due to load *W*, then the natural frequency of the free transverse vibration is

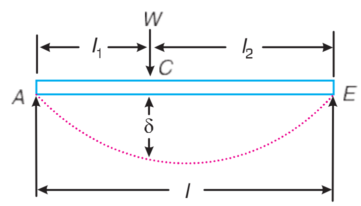
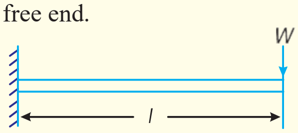
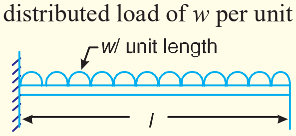
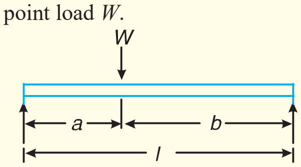
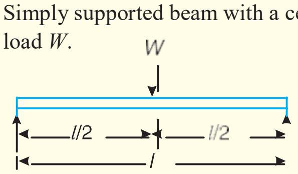


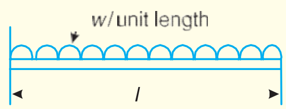
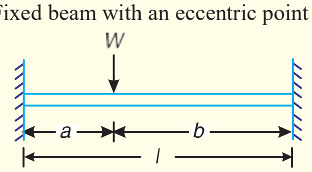
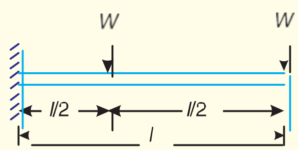
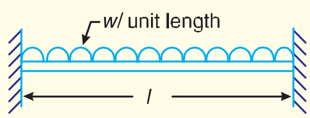

**Fig. 23.6.** Simply supported beam with a point load.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz} \qquad \ldots \text{(Substituting, } g = 9.81 \text{ m/s}^2\text{)}$$

Some of the values of the static deflection for the various types of beams and under various load conditions are given in the following table.

**Table 23.1.** Values of static deflection (δ) for the various types of beams and under various load conditions.

| *S.No.* | *Type of beam* | *Deflection* (δ) |
|---|---|---|
| 1. | Cantilever beam with a point load *W* at the free end.   | $\delta = \frac{Wl^3}{3EI}$ (at the free end) |
| 2. | Cantilever beam with a uniformly distributed load of *w* per unit length.   | $\delta = \frac{wl^4}{8EI}$ (at the free end) |
| 3. | Simply supported beam with an eccentric point load *W*.   | $\delta = \frac{Wa^2b^2}{3\,EI\,l}$ (at the point load) |
| 4. | Simply supported beam with a central point load *W*.   | $\delta = \frac{Wl^3}{48\,EI}$ (at the centre) |

| S.No. | Type of beam | Deflection ($\delta$) |
|---|---|---|
| 5. | Simply supported beam with a uniformly distributed load of $w$ per unit length.   | $\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at the centre) |
| 6. | Fixed beam with an eccentric point load $W$.   | $\delta = \frac{Wa^3b^3}{3E\,I\,l}$ (at the point load) |
| 7. | Fixed beam with a central point load $W$.   | $\delta = \frac{Wl^3}{192EI}$ (at the centre) |
| 8. | Fixed beam with a uniformly distributed load of $w$ per unit length.   | $\delta = \frac{wl^4}{384EI}$ (at the centre) |

**Example 23.2.** *A shaft of length 0.75 m, supported freely at the ends, is carrying a body of mass 90 kg at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume $E = 200$ GN/m$^2$ and shaft diameter = 50 mm.*

**Solution.** Given : $l = 0.75$ m ; $m = 90$ kg ; $a = AC = 0.25$ m ; $E = 200$ GN/m$^2$ = $200 \times 10^9$ N/m$^2$; $d = 50$ mm = 0.05 m

The shaft is shown in Fig. 23.7.

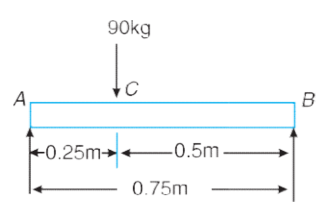


Fig. 23.7

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 \text{ m}^4$$

$$= 0.307 \times 10^{-6} \text{ m}^4$$

and static deflection at the load point (*i.e.* at point $C$),

$$\delta = \frac{Wa^2b^2}{3EI\,l} = \frac{90 \times 9.81(0.25)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 0.75} = 0.1 \times 10^{-3} \text{ m}$$

. . . ($\because b = BC = 0.5$ m)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.1 \times 10^{-3}}} = 49.85 \text{ Hz} \quad \textbf{Ans.}$$

**Example 23.3.** *A flywheel is mounted on a vertical shaft as shown in Fig. 23.8. The both ends of the shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg. Find the natural frequencies of longitudinal and transverse vibrations. Take E = 200 GN/m².*

**Solution.** Given : $d$ = 50 mm = 0.05 m ; $m$ = 500 kg ; $E$ = 200 GN/m² = 200 × $10^9$ N/m² We know that cross-sectional area of shaft,

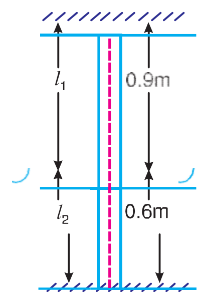


Fig. 23.8

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} \text{ m}^2$$

and moment of inertia of shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

## Natural frequency of longitudinal vibration

Let $m_1$ = Mass of flywheel carried by the length $l_1$.

∴ $m - m_1$ = Mass of flywheel carried by length $l_2$. We know that extension of length $l_1$

$$= \frac{W_1.l_1}{A.E} = \frac{m_1.g.l_1}{A.E} \quad \ldots (i)$$

Similarly, compression of length $l_2$

$$= \frac{(W - W_1)\, l_2}{A.E} = \frac{(m - m_1)\, g.l_2}{A.E} \quad \ldots (ii)$$

Since extension of length $l_1$ must be equal to compression of length $l_2$, therefore equating equations (*i*) and (*ii*),

$$m_1.l_1 = (m - m_1)\, l_2$$

$$m_1 \times 0.9 = (500 - m_1)\, 0.6 = 300 - 0.6\, m_1 \text{ or } \quad m_1 = 200 \text{ kg}$$

∴ Extension of length $l_1$,

$$\delta = \frac{m_1.g.l_1}{A.E} = \frac{200 \times 9.81 \times 0.9}{1.96 \times 10^{-3} \times 200 \times 10^9} = 4.5 \times 10^{-6} \text{ m}$$

We know that natural frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{4.5 \times 10^{-6}}} = 235 \text{ Hz} \quad \textbf{Ans.}$$

## Natural frequency of transverse vibration

We know that the static deflection for a shaft fixed at both ends and carrying a point load is given by

$$\delta = \frac{Wa^3b^3}{3E\,Il^3} = \frac{500 \times 9.81 (0.9)^3 (0.6)^3}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} (1.5)^3} = 1.24 \times 10^{-3} \text{ m}$$

. . . (Substituting $W = m.g$ ; $a = l_1$, and $b = l_2$)

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.24 \times 10^{-3}}} = 14.24 \text{ Hz} \quad \textbf{Ans.}$$

## Natural Frequency of Free Transverse Vibrations Due to Uniformly Distributed Load Acting Over a Simply Supported Shaft

Consider a shaft $AB$ carrying a uniformly distributed load of $w$ per unit length as shown in Fig. 23.9.

Let $y_1$ = Static deflection at the middle of the shaft,

$a_1$ = Amplitude of vibration at the middle of the shaft, and

$w_1$ = Uniformly distributed load per unit static deflection at the middle of the shaft = $w/y_1$.

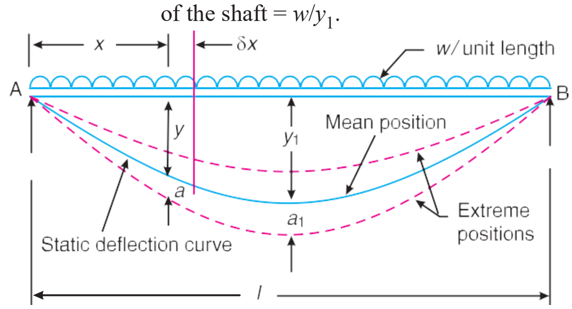



**Fig. 23.9.** Simply supported shaft carrying a uniformly distributed load.

Now, consider a small section of the shaft at a distance $x$ from $A$ and length $\delta x$.

Let $y$ = Static deflection at a distance $x$ from $A$, and

$a$ = Amplitude of its vibration.

∴ Work done on this small section

$$= \frac{1}{2} \times w_1 . a_1 . \delta x \times a = \frac{1}{2} \times \frac{w}{y_1} \times a_1 . \delta x \times a = \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a \times \delta x$$

Since the maximum potential energy at the extreme position is equal to the amount of work done to move the beam from the mean position to one of its extreme positions, therefore

Maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a . dx \qquad \ldots (i)$$

Assuming that the shape of the curve of a vibrating shaft is similar to the static deflection curve of a beam, therefore

$$\frac{a_1}{y_1} = \frac{a}{y} = \text{Constant, } C \qquad \text{or} \qquad \frac{a_1}{y_1} = C \quad \text{and } a = y.C$$

Substituting these values in equation (*i*), we have maximum potential energy at the extreme position

$$= \int_{0} \frac{1}{2} \times w \times C \times y.C.dx = \frac{1}{2} \times w.C^{2} \int_{0} y.dx \qquad \ldots (ii)$$

—

Since the maximum velocity at the mean position is $\omega.a_1$, where $\omega$ is the circular frequency of vibration, therefore

Maximum kinetic energy at the mean position

$$= \int_0^l \frac{1}{2} \times \frac{w.dx}{g} (\omega.a)^2 = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2.dx \qquad \ldots \textbf{(iii)}$$

. . .(Substituting $a = y.C$ )

We know that the maximum potential energy at the extreme position is equal to the maximum kinetic energy at the mean position, therefore equating equations **(*ii*)** and **(*iii*),**

$$\frac{1}{2} \times w \times C^2 \int_0^l y.dx = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2.dx$$

$$\therefore \qquad \omega^2 = \frac{g \int_0^l y.dx}{\int_0^l y^2.dx} \qquad \text{or} \qquad \omega = \sqrt{\frac{g \int_0^l y.dx}{\int_0^l y^2.dx}} \qquad \ldots \textbf{(iv)}$$

When the shaft is a simply supported, then the static deflection at a distance $x$ from $A$ is

$$^* y = \frac{w}{24\,EI}(x^4 - 2l\,x^3 + l^3 x) \qquad \ldots \textbf{(v)}$$

where $w$ = Uniformly distributed load unit length,

$E$ = Young's modulus for the material of the shaft, and

$I$ = Moment of inertia of the shaft.

---

* It has been proved in books on 'Strength of Materials' that maximum bending moment at a distance $x$ from $A$ is

$$(B.M.)_{max} = EI \frac{d^2 y}{dx^2} = \frac{wx^2}{2} - \frac{wl\ x}{2}$$

Integrating this expression,

$$EI.\frac{dy}{dx} = \frac{wx^3}{2 \times 3} - \frac{wl\,.\,x^2}{2 \times 2} + C_1$$

On further integrating,

$$E.I.y = \frac{wx^4}{2 \times 3 \times 4} - \frac{wl.x^3}{2 \times 2 \times 3} + C_1 x + C_2$$

$$= \frac{wx^4}{24} - \frac{wlx^3}{12} + C_1 x + C_2$$

where $C_1$ and $C_2$ are the constants of integration and may be determined from the given conditions of the problem. Here

when $x = 0, y = 0$ ; $\therefore$ $C_2 = 0$

and when $x = l, y = 0$ ; $\therefore$ $C_1 = \frac{wl^3}{24}$

Substituting the value of $C_1$, we get

$$y = \frac{w}{24\,EI}(x^4 - 2l\,x^3 + l^3 x)$$

A railway bridge.

Now integrating the above equation *(v)* within the limits from 0 to $l$,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 - 2\,lx^3 + l^3 x)\,dx = \frac{w}{24\,EI}\left[\frac{x^5}{5} - \frac{2\,lx^4}{4} + \frac{l^3 x^2}{2}\right]_0^l$$

$$= \frac{w}{24\,EI}\left[\frac{l^5}{5} - \frac{2l^5}{4} + \frac{l^5}{2}\right] = \frac{w}{24\,EI} \times \frac{l^5}{5} = \frac{w.l^5}{120\,E.I} \qquad \ldots (vi)$$

Now $$\int_0^l y^2 dx = \int_0^l \left[\frac{w}{24\,EI}(x^4 - 2l\,x^3 + l^3 x)\right]^2 dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + 4\,l^2 x^6 + l^6 x^2 - 4l\,x^7 - 4\,l^4 x^4 + 2\,l^3 x^5)\,dx$$

$$= \frac{w^2}{576\,E^2 I^2}\left[\frac{x^9}{9} + \frac{4l^2 x^7}{7} + \frac{l^6 x^3}{3} - \frac{4\,lx^8}{8} - \frac{4\,l^4 x^5}{5} + \frac{2\,l^3 x^6}{6}\right]_0^l$$

$$= \frac{w^2}{576\,E^2 I^2}\left[\frac{l^9}{9} + \frac{4l^9}{7} + \frac{l^9}{3} - \frac{4l^9}{8} - \frac{4l^9}{5} + \frac{2l^9}{6}\right]$$

$$= \frac{w^2}{576\,E^2 I^2} \times \frac{31l^9}{630} \qquad \ldots (vii)$$

Substituting the value in equation *(iv)* from equations *(vi)* and *(vii)*, we get circular frequency due to uniformly distributed load,

$$\omega = \sqrt{g\left(\frac{wl^5}{120\,EI} \times \frac{576\,E^2 I^2 \times 630}{w^2 \times 31l^9}\right)}$$

$$= \sqrt{\frac{24\,EI}{wl^4} \times \frac{630}{155}\,g} = \pi^2 \sqrt{\frac{EI\,g}{wl^4}} \qquad \text{. . . (viii)}$$

$\therefore$ Natural frequency due to uniformly distributed load,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi}\sqrt{\frac{EI\,g}{wl^4}} = \frac{\pi}{2}\sqrt{\frac{EIg}{wl^4}} \qquad \text{. . . (ix)}$$

We know that the static deflection of a simply supported shaft due to uniformly distributed load of $w$ per unit length, is

$$\delta_S = \frac{5\,wl^4}{384\,EI} \qquad \text{or} \qquad \frac{EI}{wl^4} = \frac{5}{384\,\delta_S}$$

Equation ***(ix)*** may be written as

$$f_n = \frac{\pi}{2}\sqrt{\frac{5\,g}{384\,\delta_S}} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz} \qquad \text{. . . (Substituting, } g = 9.81 \text{ m/s}^2\text{)}$$

## Natural Frequency of Free Transverse Vibrations of a Shaft Fixed at Both Ends Carrying a Uniformly Distributed Load

Consider a shaft $AB$ fixed at both ends and carrying a uniformly distributed load of $w$ per unit length as shown in Fig. 23.10.

We know that the static deflection at a distance $x$ from $A$ is given by

$$^{*}\;y = \frac{w}{24\,EI}(x^4 + l^2x^2 - 2lx^3) \qquad \text{. . . (i)}$$

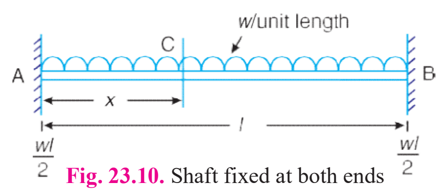



**Fig. 23.10.** Shaft fixed at both ends carrying a uniformly distributed load.

* It has been proved in books on 'Strength of Materials' that the bending moment at a distance $x$ from $A$ is

$$M = EI\frac{d^2y}{dx^2} = \frac{wl^2}{12} + \frac{wx^2}{2} - \frac{wlx}{2}$$

Integrating this equation,

$$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{2\times 3} - \frac{wlx^2}{2\times 2} + C_1$$

where $C_1$ is the constant of integration. We know that when $x = 0, \frac{dy}{dx} = 0$. Therefore $C_1 = 0$.

or $$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{6} - \frac{wlx^2}{4}$$

Integrating the above equation,

$$EI.y = \frac{wl^2x^2}{12\times 2} + \frac{wx^4}{6\times 4} - \frac{wl}{4}\times\frac{x^3}{3} + C = \frac{wl^2x^2}{24} + \frac{wx^4}{24} - \frac{wlx^3}{12} + C_2$$

where $C_2$ is the constant of integration. We know that when $x = 0, y = 0$. Therefore $C_2 = 0$.

or $$EI.y = \frac{w}{24}(l^2 x^2 + x^4 - 2lx^3)$$

or $$y = \frac{w}{24EI}(x^4 + l^2 x^2 - 2\,lx^3)$$

Integrating the above equation within limits from 0 to $l$,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 + l^2x^2 - 2l\,x^3)\,dx$$

$$= \frac{w}{24\,EI}\left[\frac{x^5}{5} + \frac{l^2x^3}{3} - \frac{2l\,x^4}{4}\right]_0^l = \frac{w}{24\,EI}\left[\frac{l^5}{5} + \frac{l^5}{3} - \frac{2l^5}{4}\right]$$

$$= \frac{w}{24\,EI}\times\frac{l^5}{30} = \frac{wl^5}{720\,EI}$$

Now integrating $y^2$ within the limits from 0 to $l$,

$$\int_0^l y^2\,dx = \left(\frac{w}{24\,EI}\right)^2\int_0^l (x^4 + l^2x^2 - 2l\,x^3)^2\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2\int_0^l (x^8 + l^4x^4 + 4l^2x^6 + 2l^2x^6 - 4l\,x^7 - 2l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2\int_0^l (x^8 + l^4x^4 + 6l^2x^6 + 4l\,x^7 - 2l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2\left[\frac{x^9}{9} + \frac{l^4x^5}{5} + \frac{6l^2x^7}{7} - \frac{4l\,x^8}{8} - \frac{2l^3x^6}{6}\right]_0^l$$

$$= \left(\frac{w}{24\,EI}\right)^2\left[\frac{l^9}{9} + \frac{l^9}{5} + \frac{6l^9}{7} - \frac{4l^9}{8} - \frac{2l^9}{6}\right] = \left(\frac{w}{24\,EI}\right)^2\frac{l^9}{630}$$

We know that

$$\omega^2 = \frac{g\int_0^l y\,dx}{\int_0^l y^2\,dx} = g\times\frac{wl^5}{720\,EI}\times\frac{(24\,EI)^2\times 630}{w^2l^9} = \frac{504\,EIg}{wl^4}$$

$\therefore$ $$\omega = \sqrt{\frac{504\,EIg}{wl^4}}$$

and natural frequency,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{504\,EIg}{wl^4}} = 3.573\sqrt{\frac{E\,I\,g}{wl^4}}$$

Since the static deflection of a shaft fixed at both ends and carrying a uniformly distributed load is

$$\delta_S = \frac{wl^4}{384\,EI} \quad\text{or}\quad \frac{E\,I}{wl^4} = \frac{1}{384\,\delta_S}$$

$\therefore$ $$f_n = 3.573\sqrt{\frac{g}{384\,\delta_S}} = \frac{0.571}{\sqrt{\delta_S}}\text{ Hz}$$ . . . (Substituting, $g = 9.81$ m/s$^2$)

## Natural Frequency of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

Consider a shaft *AB* of negligible mass loaded with point loads $W_1$, $W_2$, $W_3$ and $W_4$ etc. in newtons, as shown in Fig. 23.11. Let $m_1$, $m_2$, $m_3$ and $m_4$ etc. be the corre- sponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods :

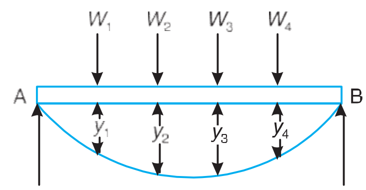



**Fig. 23.11.** Shaft carrying a number of point loads.

### 1. Energy (or Rayleigh's) method

Let $y_1$, $y_2$, $y_3$, $y_4$ etc. be total deflection under loads $W_1$, $W_2$, $W_3$ and $W_4$ etc. as shown in Fig. 23.11. We know that maximum potential energy

$$= \frac{1}{2} \times m_1.g.y_1 + \frac{1}{2} \times m_2.g.y_2 + \frac{1}{2} m_3.g.y_3 + \frac{1}{2} \times m_4.g.y_4 + .....$$

$$= \frac{1}{2} \Sigma\, m.g.y$$

and maximum kinetic energy

$$= \frac{1}{2} \times m_1 (\omega.y_1)^2 + \frac{1}{2} \times m_2 (\omega.y_2)^2 + \frac{1}{2} \times m_3 (\omega.y_3)^2 + \frac{1}{2} \times m_4 (\omega.y_4)^2 + ......$$

$$= \frac{1}{2} \times \omega^2 \left[ m_1 (y_1)^2 + m_2 (y_2)^2 + m_3 (y_3)^2 + m_4 (y_4)^2 + ..... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma\, m.y^2 \qquad \text{. . . ( where } \omega = \text{Circular frequency of vibration)}$$

Equating the maximum kinetic energy to the maximum potential energy, we have

$$\frac{1}{2} \times \omega^2\, \Sigma\, m.y^2 = \frac{1}{2} \Sigma\, m.g.y$$

$$\therefore \qquad \omega^2 = \frac{\Sigma\, m.g.y}{\Sigma\, m.y^2} = \frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2}}$$

∴ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2}}$$

### 2. Dunkerley's method

The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + .... + \frac{1}{(f_{ns})^2}$$

where $f_n$ = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

$f_{n1}$, $f_{n2}$, $f_{n3}$, etc. = Natural frequency of transverse vibration of each point load.

$f_{ns}$ = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft *AB* loaded as shown in Fig. 23.12.

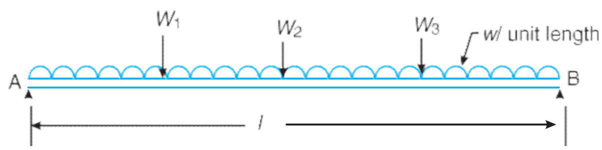



**Fig. 23.12.** Shaft carrying a number of point loads and a uniformly distributed load.

Let $\delta_1, \delta_2, \delta_3$, etc. = Static deflection due to the load $W_1$, $W_2$, $W_3$ etc. when considered separately.

$\delta_S$ = Static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load $W_1$,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load $W_2$,

$$f_{n_2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load $W_3$,

$$f_{n_3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Suspension spring of an automobile.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + \ldots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \ldots + \frac{\delta_S}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}\right]$$

or
$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + .... + \frac{\delta_S}{1.27}}} \text{ Hz}$$

**Notes : 1.** When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_S = 0$ .

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + ....}} \text{ Hz}$$

**2.** The value of $\delta_1$, $\delta_2$ , $\delta_3$ etc. for a simply supported shaft may be obtained from the relation

$$\delta = \frac{Wa^2 b^2}{3\, EIl}$$

where $\delta$ = Static deflection due to load $W$,

$a$ and $b$ = Distances of the load from the ends,

$E$ = Young's modulus for the material of the shaft,

$I$ = Moment of inertia of the shaft, and

$l$ = Total length of the shaft.

**Example 23.4.** *A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support. The Young's modulus for shaft material is 200 GN/m². Find the frequency of transverse vibration.*

**Solution.** Given : $d$ = 50 mm = 0.05 m ; $l$ = 3 m, $W_1$ = 1000 N ; $W_2$ = 1500 N ; $W_3$ = 750 N; $E$ = 200 GN/m² = 200 × $10^9$ N/m²

The shaft carrying the loads is shown in Fig. 23.13

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

and the static deflection due to a point load $W$,

$$\delta = \frac{Wa^2 b^2}{3\, EIl}$$

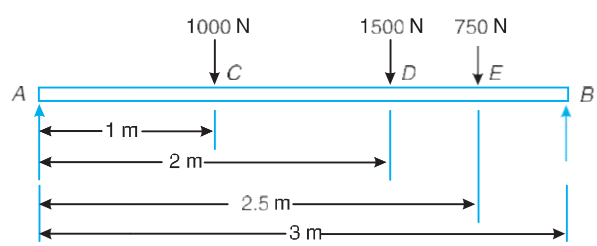



**Fig. 23.13**

$\therefore$ Static deflection due to a load of 1000 N,

$$\delta_1 = \frac{1000 \times 1^2 \times 2^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 7.24 \times 10^{-3} \text{ m}$$

. . . (Here $a$ = 1 m, and $b$ = 2 m)

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500 \times 2^2 \times 1^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 10.86 \times 10^{-3} \text{ m}$$

. . . (Here $a$ = 2 m, and $b$ = 1 m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750 (2.5)^2 (0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 2.12 \times 10^{-3} \text{ m}$$

. . . (Here $a$ = 2.5 m, and $b$ = 0.5 m)

We know that frequency of transverse vibration,

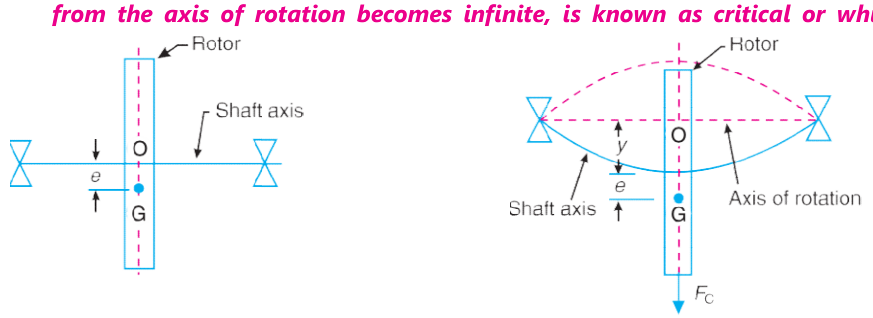
$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz}$$ **Ans.**

## Critic al or Whirling Speed of a Shaft

In actual practice, a rotating shaft carries different mountings and accessories in the form of gears, pulleys, etc. When the gears or pulleys are put on the shaft, the centre of gravity of the pulley or gear does not coincide with the centre line of the bearings or with the axis of the shaft, when the shaft is stationary. This means that the centre of gravity of the pulley or gear is at a certain distance from the axis of rotation and due to this, the shaft is subjected to centrifugal force. This force will bent the shaft which will further increase the distance of centre of gravity of the pulley or gear from the axis of rotation. This correspondingly increases the value of centrifugal force, which further increases the distance of centre of gravity from the axis of rotation. This effect is cumulative and ultimately the shaft fails. The bending of shaft not only depends upon the value of eccentricity (distance between centre of gravity of the pulley and the axis of rotation) but also depends upon the speed at which the shaft rotates.

***The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as critical or whirling speed.***




($a$) When shaft is stationary. ($b$) When shaft is rotating.

**Fig. 23.14.** Critical or whirling speed of a shaft.

Consider a shaft of negligible mass carrying a rotor, as shown in Fig.23.14 ($a$). The point $O$ is on the shaft axis and $G$ is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig. 23.14 ($b$) shows the shaft when rotating about the axis of rotation at a uniform speed of $\omega$ rad/s.

Let $m$ = Mass of the rotor,

$e$ = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,

$y$ = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at $\omega$ rad/s, and

$s$ = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.

Since the shaft is rotating at $\omega$ rad/s, therefore centrifugal force acting radially outwards through $G$ causing the shaft to deflect is given by

$$F_C = m.\omega^2 (y + e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection $y$,

$$= s.y$$

For the equilibrium position,

$$m.\omega^2 (y + e) = s.y$$

or
$$m.\omega^2.y + m.\omega^2.e = s.y \quad \text{or} \quad y(s - m.\omega^2) = m.\omega^2.e$$

$$\therefore \quad y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{s/m - \omega^2} \qquad \ldots (i)$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \quad \text{or} \quad y = \frac{\omega^2.e}{(\omega_n)^2 - \omega^2} \qquad \ldots \text{[ From equation } (i) \text{ ]}$$

A little consideration will show that when $\omega > \omega_n$, the value of $y$ will be negative and the shaft deflects is the opposite direction as shown dotted in Fig 23.14 (*b*).

In order to have the value of $y$ always positive, both ***plus*** and ***minus*** signs are taken.

$$\therefore \quad y = \pm \frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$ )

We see from the above expression that when $\omega_n = \omega_c$, the value of $y$ becomes infinite.

Therefore $\omega_c$ is the **critical or whirling speed.**

$\therefore$ Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \qquad \ldots \left(\because \delta = \frac{m.g}{s}\right)$$

If $N_c$ is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

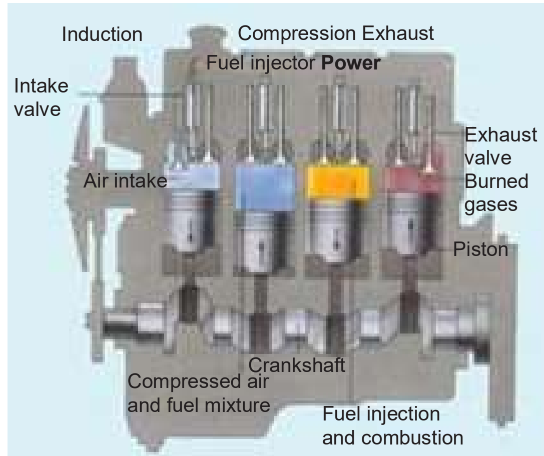
where $\delta$ = Static deflection of the shaft in metres.

*Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.***

**Notes : 1.** When the centre of gravity of the rotor lies between the centre line of the shaft and the centre line of the bearing, $e$ is taken negative. On the other hand, if the centre of gravity of the rotor does not lie between the centre line of the shaft and the centre line of the bearing (as in the above article) the value of $e$ is taken positive.

**2.** To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s. The Dunkerley's method may be used for calculating the frequency.

**3.** A shaft supported is short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.




Diesel engines have several advantages over petrol engines. They do not need an electrical ignition system; they use cheaper fuel; and they do not need a carburettor. Diesel engines also have a greater ability to convert the stored energy in the fuel into mechanical energy, or work.

Note : This picture is given as additional information and is not a direct example of the current chapter.

**Example 23.5.** *Calculate the whirling speed of a shaft 20 mm diameter and 0.6 m long carrying a mass of 1 kg at its mid-point. The density of the shaft material is 40 Mg/m³, and Young's modulus is 200 GN/m². Assume the shaft to be freely supported.*

**Solution.** Given : $d = 20$ mm $= 0.02$ m ; $l = 0.6$ m ; $m_1 = 1$ kg ; $\rho = 40$ Mg/m³ $= 40 \times 10^6$ g/m³ $= 40 \times 10^3$ kg/m³ ; $E = 200$ GN/m² $= 200 \times 10^9$ N/m²

The shaft is shown in Fig. 23.15.

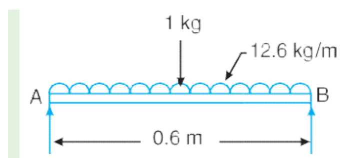



**Fig. 23.15**

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.02)^4 \text{ m}^4$$

$$= 7.855 \times 10^{-9} \text{ m}^4$$

Since the density of shaft material is $40 \times 10^3$ kg/m³, therefore mass of the shaft per metre length,

$$m = \text{Area} \times \text{length} \times \text{density} = \frac{\pi}{4}(0.02)^2 \times 1 \times 40 \times 10^3 \text{ s} = 12.6 \text{ kg/m}$$

We know that static deflection due to 1 kg of mass at the centre,

$$\delta = \frac{Wl^3}{48\,EI} = \frac{1 \times 9.81(0.6)^3}{48 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 28 \times 10^{-6} \text{ m}$$

and static deflection due to mass of the shaft,

$$\delta_S = \frac{5\,wl^4}{384\,EI} = \frac{5 \times 12.6 \times 9.81(0.6)^4}{384 \times 200 \times 10^9 \times 7.855 \times 10^{-9}} = 0.133 \times 10^{-3} \text{ m}$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta + \dfrac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{28\times10^{-6} + \dfrac{0.133\times10^{-3}}{1.27}}}$$

$$= \frac{0.4985}{11.52\times10^{-3}} = 43.3 \text{ Hz}$$

Let $N_c$ = Whirling speed of a shaft.

We know that whirling speed of a shaft in r.p.s. is equal to the frequency of transverse vibration in Hz , therefore

$$N_c = 43.3 \text{ r.p.s.} = 43.3 \times 60 = 2598 \text{ r.p.m.}$$ **Ans.**

**Example 23.6.** *A shaft 1.5 m long, supported in flexible bearings at the ends carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 375 mm from the centre towards left. The shaft is hollow of external diameter 75 mm and internal diameter 40 mm. The density of the shaft material is 7700 kg/m³ and its modulus of elasticity is 200 GN/m². Find the lowest whirling speed of the shaft, taking into account the mass of the shaft.*

**Solution.** $l$ = 1.5 m ; $m_1 = m_2$ = 50 kg ; $d_1$ = 75 mm = 0.075 m ; $d_2$ = 40 mm = 0.04 m ; $\rho$ = 7700 kg/m³ ; $E$ = 200 GN/m² = 200 × $10^9$ N/m²

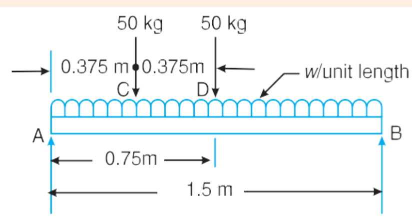


Fig. 23.16

The shaft is shown in Fig. 23.16.

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64}\left[(d_1)^4 - (d_2)^4\right] = \frac{\pi}{64}\left[(0.075)^4 - (0.04)^4\right] = 1.4\times10^{-6} \text{ m}^4$$

Since the density of shaft material is 7700 kg/m³, therefore mass of the shaft per metre length,

$$m_S = \text{Area} \times \text{length} \times \text{density}$$

$$= \frac{\pi}{4}\left[(0.075)^2 - (0.04)^2\right] 1 \times 7700 = 24.34 \text{ kg/m}$$

We know that the static deflection due to a load $W$

$$= \frac{Wa^2b^2}{3\,EIl} = \frac{m.ga^2b^2}{3\,EIl}$$

$\therefore$ Static deflection due to a mass of 50 kg at $C$,

$$\delta_1 = \frac{m_1 g a^2 b^2}{3\,EIl} = \frac{50 \times 9.81(0.375)^2\,(1.125)^2}{3\times 200\times10^9 \times 1.4\times10^{-6} \times 1.5} = 70 \times 10^{-6} \text{ m}$$

. . . (Here $a$ = 0.375 m, and $b$ = 1.125 m)

Similarly, static deflection due to a mass of 50 kg at $D$

$$\delta_2 = \frac{m_1 g a^2 b^2}{3\,EIl} = \frac{50 \times 9.81(0.75)^2\,(0.75)^2}{3\times 200\times10^9 \times 1.4\times10^{-6} \times 1.5} = 123 \times 10^{-6} \text{ m}$$

. . . (Here $a = b$ = 0.75 m)

We know that static deflection due to uniformly distributed load or mass of the shaft,

$$\delta_S = \frac{5}{384} \times \frac{wl^4}{EI} = \frac{5}{384} \times \frac{24.34 \times 9.81(1.5)^4}{200 \times 10^9 \times 1.4 \times 10^{-6}} = 56 \times 10^{-6} \text{ m}$$

. . . (Substituting, $w = m_S \times g$)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \frac{\delta_S}{1.27}}} = \frac{0.4985}{\sqrt{70 \times 10^{-6} + 123 \times 10^{-6} + \frac{56 \times 10^{-6}}{1.27}}} \text{ Hz}$$

$= 32.4$ Hz

Since the whirling speed of shaft ($N_c$) in r.p.s. is equal to the frequency of transverse vibration in Hz, therefore

$N_c = 32.4$ r.p.s. $= 32.4 \times 60 = 1944$ r.p.m. **Ans.**

**Example 23.7.** *A vertical shaft of 5 mm diameter is 200 mm long and is supported in long bearings at its ends. A disc of mass 50 kg is attached to the centre of the shaft. Neglecting any increase in stiffness due to the attachment of the disc to the shaft, find the critical speed of rotation and the maximum bending stress when the shaft is rotating at 75% of the critical speed. The centre of the disc is 0.25 mm from the geometric axis of the shaft. E = 200 GN/m².*

**Solution.** Given : $d = 5$ mm $= 0.005$ m ; $l = 200$ mm $= 0.2$ m ; $m = 50$ kg ; $e = 0.25$ mm $= 0.25 \times 10^{-3}$ m ; $E = 200$ GN/m$^2$ $= 200 \times 10^9$ N/m$^2$

## Critical speed of rotation

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.005)^4 = 30.7 \times 10^{-12} \text{ m}^4$$

Since the shaft is supported in long bearings, it is assumed to be fixed at both ends. We know that the static deflection at the centre of the shaft due to a mass of 50 kg,

$$\delta = \frac{Wl^3}{192\, EI} = \frac{50 \times 9.81(0.2)^3}{192 \times 200 \times 10^9 \times 30.7 \times 10^{-12}} = 3.33 \times 10^{-3} \text{ m}$$

. . . ($\because W = m.g$)

We know that critical speed of rotation (or natural frequency of transverse vibrations),

$$N_c = \frac{0.4985}{\sqrt{3.33 \times 10^{-3}}} = 8.64 \text{ r.p.s.}$$ **Ans.**

## Maximum bending stress

Let $\sigma$ = Maximum bending stress in N/m$^2$, and

$N$ = Speed of the shaft = 75% of critical speed = $0.75\, N_c$ . . . (Given)

When the shaft starts rotating, the additional dynamic load ($W_1$) to which the shaft is subjected, may be obtained by using the bending equation,

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma . I}{y_1}$$

We know that for a shaft fixed at both ends and carrying a point load ($W_1$) at the centre, the maximum bending moment

$$M = \frac{W_1 . l}{8}$$

$$\therefore \quad \frac{W_1 . l}{8} = \frac{\sigma \cdot I}{d/2} \qquad \ldots (\because y_1 = d/2)$$

and

$$W_1 = \frac{\sigma . I}{d/2} \times \frac{8}{l} = \frac{\sigma \times 30.7 \times 10^{-12}}{0.005/2} \times \frac{8}{0.2} = 0.49 \times 10^{-6} \sigma \text{ N}$$

$\therefore$ Additional deflection due to load $W_1$,

$$y = \frac{W_1}{W} \times \delta = \frac{0.49 \times 10^{-6} \sigma}{50 \times 9.81} \times 3.33 \times 10^{-3} = 3.327 \times 10^{-12} \sigma$$

We know that

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{N_c}{N}\right)^2 - 1} \qquad \ldots \text{(Substituting } \omega = N_c \text{ and } \omega = N_c)$$

$$3.327 \times 10^{-12} \sigma = \frac{\pm 0.25 \times 10^{-3}}{\left(\frac{N_c}{0.75 N_c}\right)^2 - 1} = \pm 0.32 \times 10^{-3}$$

$$\sigma = 0.32 \times 10^{-3} / 3.327 \times 10^{-12} = 0.0962 \times 10^{9} \text{ N/m}^2 \qquad \ldots \text{( Taking + ve sign )}$$

$$= 96.2 \times 10^6 \text{ N/m}^2 = 96.2 \text{ MN/m}^2$$ **Ans.**

**Example 23.8.** *A vertical steel shaft 15 mm diameter is held in long bearings 1 metre apart and carries at its middle a disc of mass 15 kg. The eccentricity of the centre of gravity of the disc from the centre of the rotor is 0.30 mm.*

*The modulus of elasticity for the shaft material is 200 GN/$m^2$ and the permissible stress is 70 MN/$m^2$. Determine :* **1.** *The critical speed of the shaft and* **2.** *The range of speed over which it is unsafe to run the shaft. Neglect the mass of the shaft.*

[*For a shaft with fixed end carrying a concentrated load (W) at the centre assume* $\delta = \frac{Wl^3}{192\, EI}$, *and* $M = \frac{W . l}{8}$, *where* $\delta$ *and M are maximum deflection and bending moment respectively*].

**Solution.** Given : $d$ = 15 mm = 0.015 m ; $l$ = 1 m ; $m$ = 15 kg ; $e$ = 0.3 mm = $0.3 \times 10^{-3}$ m ; $E$ = 200 GN/$m^2$ = $200 \times 10^9$ N/$m^2$ ; $\sigma$ = 70 MN/$m^2$ = $70 \times 10^6$ N/$m^2$

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64} (0.015)^4 = 2.5 \times 10^{-9} \text{ m}^4$$

## 1. Critical speed of the shaft

Since the shaft is held in long bearings, therefore it is assumed to be fixed at both ends. We know that the static deflection at the centre of shaft,

$$\delta = \frac{Wl^3}{192\, EI} = \frac{15 \times 9.81 \times 1^3}{192 \times 200 \times 10^9 \times 2.5 \times 10^{-9}} = 1.5 \times 10^{-3} \text{ m} \qquad \ldots (\because W = m.g)$$

$\therefore$ Natural frequency of transverse vibrations,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{1.5 \times 10^{-3}}} = 12.88 \text{ Hz}$$

We know that the critical speed of the shaft in r.p.s. is equal to the natural frequency of transverse vibrations in Hz.

$\therefore$ Critical speed of the shaft,

$$N_c = 12.88 \text{ r.p.s.} = 12.88 \times 60 = 772.8 \text{ r.p.m. } \textbf{Ans.}$$

## 2. Range of speed

Let $N_1$ and $N_2$ = Minimum and maximum speed respectively.

When the shaft starts rotating, the additional dynamic load ($W_1 = m_1.g$) to which the shaft is subjected may be obtained from the relation

$$\frac{M}{I} = \frac{\sigma}{y_1} \quad \text{or} \quad M = \frac{\sigma.I}{y_1}$$

Since $M = \frac{W_1.l}{8} = \frac{m_1.g.l}{8}$, and $y_1 = \frac{d}{2}$, therefore

$$\frac{m_1.g.l}{8} = \frac{\sigma.I}{d/2}$$

or

$$m_1 = \frac{8 \times 2 \times \sigma \times I}{d.g.l} = \frac{8 \times 2 \times 70 \times 10^6 \times 2.5 \times 10^{-9}}{0.015 \times 9.81 \times 1} = 19 \text{ kg}$$

$\therefore$ Additional deflection due to load $W_1 = m_1 g$,

$$y = \frac{W_1}{W} \times \delta = \frac{m_1}{m} \times \delta = \frac{19}{15} \times 1.5 \times 10^{-3} = 1.9 \times 10^{-3} \text{ m}$$

We know that,

$$y = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1} \quad \text{or} \quad \pm\frac{y}{e} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1}$$

... (Substituting, $\omega_c = N_c$, and $\omega = N$)

$$\therefore \quad \pm\frac{1.9 \times 10^{-3}}{0.3 \times 10^{-3}} = \frac{1}{\left(\frac{N_c}{N}\right)^2 - 1} \quad \text{or} \quad \left(\frac{N_c}{N}\right)^2 - 1 = \pm\frac{0.3}{1.9} = \pm 0.16$$

$$\left(\frac{N_c}{N}\right)^2 = 1 \pm 0.16 = 1.16 \text{ or } 0.84$$

... (Taking first plus sign and then negative sign)

or

$$N = \frac{N_c}{\sqrt{1.16}} \quad \text{or} \quad \frac{N_c}{\sqrt{0.84}}$$

$$\therefore \quad N_1 = \frac{N_c}{\sqrt{1.16}} = \frac{772.8}{\sqrt{1.16}} = 718 \text{ r.p.m.}$$

and

$$N_2 = \frac{N_c}{\sqrt{0.84}} = \frac{772.8}{\sqrt{0.84}} = 843 \text{ r.p.m.}$$

Hence the range of speed is from 718 r.p.m. to 843 r.p.m. **Ans.**

## Frequency of Free Damped Vibrations (Viscous Damping)

We have already discussed that the motion of a body is resisted by frictional forces. In vibrating systems, the effect of friction is referred to as damping. The damping provided by fluid resistance is known as ***viscous damping.***

We have also discussed that in damped vibrations, the amplitude of the resulting vibration gradually diminishes. This is due to the reason that a certain amount of energy is always dissipated to overcome the frictional resistance. The resistance to the motion of the body is provided partly by the medium in which the vibration takes place and partly by the internal friction, and in some cases partly by a dash pot or other external damping device.

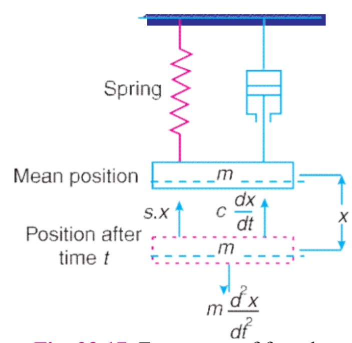



**Fig. 23.17.** Frequency of free damped vibrations.

Consider a vibrating system, as shown in Fig. 23.17, in which a mass is suspended from one end of the spiral spring and the other end of which is fixed. A damper is provided between the mass and the rigid support.

Let $m$ = Mass suspended from the spring,

$s$ = Stiffness of the spring,

$x$ = Displacement of the mass from the mean position at time $t$,

$\delta$ = Static deflection of the spring $= m.g/s$, and

$c$ = Damping coefficient or the damping force per unit velocity.

Since in viscous damping, it is assumed that the frictional resistance to the motion of the body is directly proportional to the speed of the movement, therefore

Damping force or frictional force on the mass acting in ***opposite*** direction to the motion of the mass

$$= c \times \frac{dx}{dt}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= m \times \frac{d^2x}{dt^2}$$

Riveting Machine

Note : This picture is given as additional information and is not a direct example of the current chapter.

and spring force on the mass, acting in ***opposite*** direction to the motion of the mass,

$$= s.x$$

Therefore the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -\left(c \times \frac{dx}{dt} + s.x\right)$$

...(Negative sign indicates that the force opposes the motion)

or $$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = 0$$

or $$^*\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

This is a differential equation of the second order. Assuming a solution of the form $x = e^{kt}$ where $k$ is a constant to be determined. Now the above differential equation reduces to

$$k^2.e^{kt} + \frac{c}{m} \times k.e^{kt} + \frac{s}{m} \times e^{kt} = 0 \qquad \ldots \left[\because \frac{dx}{dt} = ke^{kt}, \text{ and } \frac{d^2x}{dt^2} = k^2.e^{kt}\right]$$

or $$k^2 + \frac{c}{m} \times k + \frac{s}{m} = 0 \qquad \ldots (i)$$

and $$k = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4 \times \frac{s}{m}}}{2}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

$\therefore$ The two roots of the equation are

$$k_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

and $$k_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The most general solution of the differential equation ***(i)*** with its right hand side equal to zero has only complementary function and it is given by

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t} \qquad \ldots (ii)$$

where $C_1$ and $C_2$ are two arbitrary constants which are to be determined from the initial conditions of the motion of the mass.

It may be noted that the roots $k_1$ and $k_2$ may be real, complex conjugate (imaginary) or equal. We shall now discuss these three cases as below :

* A system described by this equation is said to be a single degree of freedom harmonic oscillator with viscous damping.

## 1. When the roots are real (overdamping)

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$, then the roots $k_1$ and $k_2$ are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic***. When the roots are real, the most general solution of the differential equation is

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

$$= C_1 e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t} + C_2 e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t}$$

**Note :** In actual practice, the overdamped vibrations are avoided.

## 2. When the roots are complex conjugate (underdamping)

If $\frac{s}{m} > \left(\frac{c}{2m}\right)^2$, then the radical (*i.e.* the term under the square root) becomes negative.

The two roots $k_1$ and $k_2$ are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping***. The two roots are

$$k_1 = -\frac{c}{2m} + i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

and
$$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

where $i$ is a Greek letter known as iota and its value is $\sqrt{-1}$. For the sake of mathematical calculations, let

$$\frac{c}{2m} = a;\quad \frac{s}{m} = (\omega_n)^2;\ \text{and}\quad \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{(\omega_n)^2 - a^2}$$

Therefore the two roots may be written as

$$k_1 = -a + i\,\omega_d \;;\quad \text{and}\quad k_2 = -a - i\,\omega_d$$

We know that the general solution of a differential equation is

$$x = C_1\,e^{k_1 t} + C_2\,e^{k_2 t} = C_1\,e^{(-a+i\omega_d)t} + C_2\,e^{(-a-i\omega_d)t}$$

$$= e^{-at}\left(C_1\,e^{i\omega_d . t} + C_2\,e^{-i\omega_d t}\right) \quad \ldots(\text{Using } e^{m+n} = e^m \times e^n) \ \ldots(iii)$$

Now according to Euler's theorem

$$e^{+i\,\theta} = \cos\theta + i\sin\theta \;;\ \text{and}\ e^{-i\,\theta} = \cos\theta - i\sin\theta$$

Therefore the equation ***(iii)*** may be written as

$$x = e^{-at}\left[C_1(\cos\omega_d .t + i\sin\omega_d .t) + C_2(\cos\omega_d .t - i\sin\omega_d .t)\right]$$

$$= e^{-at}\left[(C_1 + C_2)\cos\omega_d .t + i\,(C_1 - C_2)\sin\omega_d .t\right]$$

Let
$$C_1 + C_2 = A,\ \text{and}\ i\,(C_1 - C_2) = B$$

$\therefore \quad x = e^{-at}(A\cos\omega_d.t + B\sin\omega_d.t)$ . . . (*iv*)

Again, let $A = C\cos\theta$ , and $B = C\sin\theta$ , therefore

$$C = \sqrt{A^2 + B^2} \text{ , and } \tan\theta = \frac{B}{A}$$

Now the equation (*iv*) becomes

$$x = e^{-at}(C\cos\theta\cos\omega_d.t + C\sin\theta\sin\omega_d.t)$$

$$= Ce^{-at}\cos(\omega_d.t - \theta) \quad \text{. . . (v)}$$

If $t$ is measured from the instant at which the mass $m$ is released after an initial displace- ment $A$, then

$A = C\cos\theta$ . . . [Substituting $x = A$ and $t = 0$ in equation (*v*)]

and when $\theta = 0$ , then $A = C$

$\therefore$ The equation (*v*) may be written as

$$x = Ae^{-at}\cos\omega_d.t \quad \text{. . . (vi)}$$

where $\omega_d = \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{(\omega_n)^2 - a^2}$ ; and $a = \frac{c}{2m}$

We see from equation (*vi*), that the motion of the mass is simple harmonic whose circular damped frequency is $\omega_d$ and the amplitude is $Ae^{-at}$ which diminishes exponentially with time as shown in Fig. 23.18. Though the mass eventually returns to its equilibrium position because of its inertia, yet it overshoots and the oscillations may take some considerable time to die away.

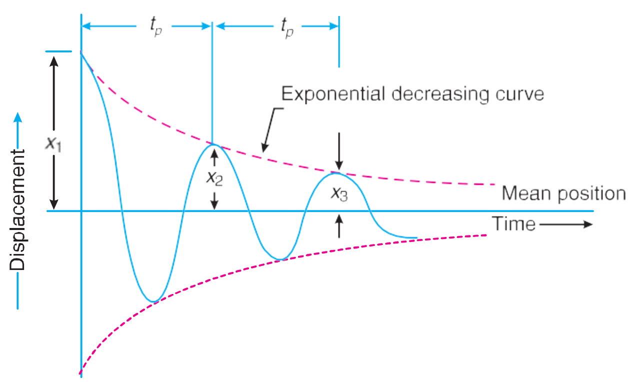



**Fig. 23.18.** Underdamping or small damping.

We know that the periodic time of vibration,

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

and frequency of damped vibration,

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} \quad \text{. . . (vii)}$$

**Note :** When no damper is provided in the system, then $c = 0$. Therefore the frequency of the undamped vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

. . . [Substituting $c = 0$, in equation (*vii*)]

It is the same as discussed under free vibrations.

In a disc brake, hydraulic pressure forces friction pads to squeeze a metal disc that rotates on the same axle as the wheel. Here a disc brake is being tested.

Note : This picture is given as additional information and is not a direct example of the current chapter.

**3. When the roots are equal (critical damping).**

If $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$, then the radical becomes zero and the two roots $k_1$ and $k_2$ are equal. This is a case of **critical damping.** In other words, the critical damping is said to occur when frequency of damped vibration ($f_d$) is zero (*i.e.* motion is aperiodic). This type of damping is also avoided because the mass moves back rapidly to its equilibrium position, in the shortest possible time.

For critical damping, equation (*ii*) may be written as

$$x = (C_1 + C_2)\, e^{-\frac{c}{2m}t} = (C_1 + C_2)\, e^{-\omega_n t} \quad \ldots \left[\because \frac{c}{2m} = \sqrt{\frac{s}{m}} = \omega_n\right]$$

Thus the motion is again aperiodic. The critical damping coefficient ($c_c$) may be obtained by substituting $c_c$ for $c$ in the condition for critical damping, *i.e.*

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m} \quad \text{or} \quad c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n$$

The critical damping coefficient is the amount of damping required for a system to be critically damped.

## Damping Factor or Damping Ratio

The ratio of the actual damping coefficient ($c$) to the critical damping coefficient ($c_c$) is known as ***damping factor* or *damping ratio*.** Mathematically,

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n} \quad \ldots (\because c_c = 2\pi.\omega_n)$$

The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system.

## Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor. The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position. If $x_1$ and $x_2$ are successive values of the amplitude on the same side of the mean position, as shown in Fig. 23.18, then amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{Ae^{-at}}{Ae^{-a(t+t_p)}} = e^{a t_p} = \text{constant}$$

where $t_p$ is the period of forced oscillation or the time difference between two consecutive amplitudes. As per definition, logarithmic decrement,

$$\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{a t_p}$$

or

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a.t_p = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\ldots \left[\because \omega_d = \sqrt{(\omega_n)^2 - a^2}\right]$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\sqrt{(\omega_n)^2 - \left(\frac{c}{2m}\right)^2}} \qquad \ldots \left(\because a = \frac{c}{2m}\right)$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\omega_n\sqrt{1 - \left(\frac{c}{2m.\omega_n}\right)^2}} = \frac{c \times 2\pi}{c_c\sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \qquad \ldots (\because c_c = 2m.\omega_n)$$

$$= \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

In general, amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots = \frac{x_n}{x_{n+1}} = e^{a t_p} = \text{constant}$$

$\therefore$ Logarithmic decrement,

$$\delta = \log_e\left(\frac{x_n}{x_{n+1}}\right) = a.t_p = \frac{2\pi \times c}{\sqrt{(c_c)^2 - c^2}}$$

**Example 23.9.** *A vibrating system consists of a mass of 200 kg, a spring of stiffness 80 N/mm and a damper with damping coefficient of 800 N/m/s. Determine the frequency of vibration of the system.*

**Solution.** Given : $m = 200$ kg ; $s = 80$ N/mm $= 80 \times 10^3$ N/m ; $c = 800$ N/m/s We know that circular frequency of undamped vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{80 \times 10^3}{200}} = 20 \text{ rad/s}$$

and circular frequency of damped vibrations,

$$\omega_d = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(\omega_n)^2 - (c/2m)^2} \qquad \ldots (\because a = c/2m)$$

$$= \sqrt{(20)^2 - (800/2\times 200)^2} = 19.9 \text{ rad/s}$$

∴ Frequency of vibration of the system,

$$f_d = \omega_d / 2\pi = 19.9 / 2\pi = 3.17 \text{ Hz } \textbf{Ans.}$$

**Example 23.10.** *The following data are given for a vibratory system with viscous damping:*

*Mass = 2.5 kg ; spring constant = 3 N/mm and the amplitude decreases to 0.25 of the initial value after five consecutive cycles.*

*Determine the damping coefficient of the damper in the system.*

**Solution.** Given : $m = 2.5$ kg ; $s = 3$ N/mm = 3000 N/m ; $x_6 = 0.25\, x_1$

We know that natural circular frequency of vibration,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{3000}{2.5}} = 34.64 \text{ rad/s}$$

Let $c$ = Damping coefficient of the damper in N/m/s,

$x_1$ = Initial amplitude, and

$x_6$ = Final amplitude after five consecutive cycles = $0.25\, x_1$ …(Given)

We know that

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}$$

or

$$\frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5$$

∴

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\, x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e(1.32) = a \times \frac{2\pi}{\sqrt{(34.64)^2 - a^2}} \quad \text{or} \quad 0.2776 = \frac{a \times 2\pi}{\sqrt{1200 - a^2}}$$

Squaring both sides,

$$0.077 = \frac{39.5\, a^2}{1200 - a^2} \quad \text{or} \quad 92.4 - 0.077\, a^2 = 39.5\, a^2$$

∴ $a^2 = 2.335$ or $a = 1.53$

We know that $a = c/2m$ or $c = a \times 2m = 1.53 \times 2 \times 2.5 = 7.65$ N/m/s **Ans.**

**Example 23.11.** *An instrument vibrates with a frequency of 1 Hz when there is no damping. When the damping is provided, the frequency of damped vibrations was observed to be 0.9 Hz. Find* ***1.*** *the damping factor, and* ***2.*** *logarithmic decrement.*

Guitar

**Solution.** Given : $f_n = 1$ Hz ; $f_d = 0.9$ Hz

### 1. Damping factor

Let $m$ = Mass of the instrument in kg,

$c$ = Damping coefficient or damping force per unit velocity in N/m/s, and

$c_c$ = Critical damping coefficient in N/m/s.

We know that natural circular frequency of undamped vibrations,

$$\omega_n = 2\pi \times f_n = 2\pi \times 1 = 6.284 \text{ rad/s}$$

and circular frequency of damped vibrations,

$$\omega_d = 2\pi \times f_d = 2\pi \times 0.9 = 5.66 \text{ rad/s}$$

We also know that circular frequency of damped vibrations ($\omega_d$),

$$5.66 = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(6.284)^2 - a^2}$$

Squaring both sides,

$$(5.66)^2 = (6.284)^2 - a^2 \text{ or } 32 = 39.5 - a^2$$

$\therefore$ $a^2 = 7.5$ or $a = 2.74$

We know that, $a = c/2m$ or $c = a \times 2m = 2.74 \times 2m = 5.48\ m$ N/m/s

and $c_c = 2m.\omega_n = 2m \times 6.284 = 12.568\ m$ N/m/s

$\therefore$ Damping factor,

$$c / c_c = 5.48m / 12.568\ m = 0.436 \text{ **Ans.**}$$

### 2. Logarithmic decrement

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 5.48\,m}{\sqrt{(12.568\,m)^2 - (5.48\,m)^2}} = \frac{34.4}{11.3} = 3.04 \text{ **Ans.**}$$

**Example 23.12.** *The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness 5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find :* ***1.*** *critical damping coefficient,* ***2.*** *damping factor,* ***3.*** *logarithmic decrement, and* ***4.*** *ratio of two consecutive amplitudes.*

**Solution.** Given : $m = 8$ kg ; $s = 5.4$ N/mm = 5400 N/m

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s , therefore Damping coefficient (actual),

$$c = 40 \text{ N/m/s}$$

**1. Critical damping coefficient**

We know that critical damping coefficient,

$$c_c = 2m.\omega_n = 2m \times \sqrt{\frac{s}{m}} = 2 \times 8 \sqrt{\frac{5400}{8}} = 416 \text{ N/m/s } \textbf{Ans.}$$

**2. Damping factor**

We know that damping factor

$$= \frac{c}{c_c} = \frac{40}{416} = 0.096 \textbf{ Ans.}$$

**3. Logarithmic decrement**

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 40}{\sqrt{(416)^2 - (40)^2}} = 0.6 \textbf{ Ans.}$$

**4. Ratio of two consecutive amplitudes**

Let $x_n$ and $x_{n+1}$ = Magnitude of two consecutive amplitudes, We know that logarithmic decrement,

$$\delta = \log_e \left[\frac{x_n}{x_{n+1}}\right] \text{ or } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82 \textbf{ Ans.}$$

**Example 23.13.** *A mass suspended from a helical spring vibrates in a viscous fluid medium whose resistance varies directly with the speed. It is observed that the frequency of damped vibration is 90 per minute and that the amplitude decreases to 20 % of its initial value in one complete vibration. Find the frequency of the free undamped vibration of the system.*

Helical spring suspension of a two-wheeler.

Note : This picture is given as additional information and is not a direct example of the current chapter.

**Solution.** Given : $f_d$ = 90/min = 90/60 = 1.5 Hz We know that time period,

$$t_p = 1/f_d = 1/1.5 = 0.67 \text{ s}$$

Let $x_1$ = Initial amplitude, and

$x_2$ = Final amplitude after one complete vibration

$= 20\% \, x_1 = 0.2 \, x_1$ . . . (Given)

We know that

$$\log_e \left(\frac{x_1}{x_2}\right) = a.t_p \quad \text{or} \quad \log_e \left(\frac{x_1}{0.2\,x_1}\right) = a \times 0.67$$

$\therefore \quad \log_e 5 = 0.67\,a$ or $1.61 = 0.67\,a$ or $a = 2.4$ . . . ($\because \log_e 5 = 1.61$)

We also know that frequency of free damped vibration,

$$f_d = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2}$$

or $$(\omega_n)^2 = (2\pi \times f_d)^2 + a^2 \quad \text{. . . (By squaring and arranging)}$$

$$= (2\pi \times 1.5)^2 + (2.4)^2 = 94.6$$

$\therefore$ $$\omega_n = 9.726 \text{ rad/s}$$

We know that frequency of undamped vibration,

$$f_n = \frac{\omega_n}{2\pi} = \frac{9.726}{2\pi} = 1.55 \text{ Hz} \ \textbf{Ans.}$$

**Example 23.14.** *A coil of spring stiffness 4 N/mm supports vertically a mass of 20 kg at the free end. The motion is resisted by the oil dashpot. It is found that the amplitude at the beginning of the fourth cycle is 0.8 times the amplitude of the previous vibration. Determine the damping force per unit velocity. Also find the ratio of the frequency of damped and undamped vibrations.*

**Solution.** Given : $s$ = 4 N/mm = 4000 N/m ; $m$ = 20 kg

## Damping force per unit velocity

Let $c$ = Damping force in newtons per unit velocity *i.e.* in N/m/s

$x_n$ = Amplitude at the beginning of the third cycle,

$x_{n+1}$ = Amplitude at the beginning of the fourth cycle = 0.8 $x_n$ . . . (Given)

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{4000}{20}} = 14.14 \text{ rad/s}$$

and $$\log_e\left(\frac{x_n}{x_{n+1}}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

or $$\log_e\left(\frac{x_n}{0.8\,x_n}\right) = a \times \frac{2\pi}{\sqrt{(14.14)^2 - a^2}}$$

$$\log_e 1.25 = a \times \frac{2\pi}{\sqrt{200 - a^2}} \quad \text{or} \quad 0.223 = a \times \frac{2\pi}{\sqrt{200 - a^2}}$$

Squaring both sides

$$0.05 = \frac{a^2 \times 4\pi^2}{200 - a^2} = \frac{\quad}{200 - a^2} \qquad a^2$$

$$0.05 \times 200 - 0.05\,a^2 = 39.5a^2 \quad \text{or} \quad 39.55\,a^2 = 10$$

$\therefore$ $$a^2 = 10 / 39.55 = 0.25 \quad \text{or} \quad a = 0.5$$

We know that $$a = c / 2m$$

$\therefore$ $$c = a \times 2m = 0.5 \times 2 \times 20 = 20 \text{ N/m/s} \ \textbf{Ans.}$$

## Ratio of the frequencies

Let $$f_{n1} = \text{Frequency of damped vibrations} = \frac{\omega_d}{2\pi}$$

$$f_{n2} = \text{Frequency of undamped vibrations} = \frac{\omega_n}{2\pi}$$

$\therefore$

$$\frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \sqrt{\frac{(\omega_n)^2 - a^2}{\omega_n}} = \sqrt{\frac{(14.14)^2 - (0.5)^2}{14.14}}$$

$$\ldots \left(\because\ \omega_d = \sqrt{(\omega_n)^2 - a^2}\right)$$

$= 0.999$ **Ans.**

**Example 23.15.** *A machine of mass 75 kg is mounted on springs and is fitted with a dashpot to damp out vibrations. There are three springs each of stiffness 10 N/mm and it is found that the amplitude of vibration diminishes from 38.4 mm to 6.4 mm in two complete oscillations. Assuming that the damping force varies as the velocity, determine :* ***1.*** *the resistance of the dashpot at unit velocity ;* ***2.*** *the ratio of the frequency of the damped vibration to the frequency of the undamped vibration ; and* ***3.*** *the periodic time of the damped vibration.*

**Solution.** Given : $m = 75$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_1 = 38.4$ mm $= 0.0384$ m ; $x_3 = 6.4$ mm $= 0.0064$ m

Since the stiffness of each spring is $10 \times 10^3$ N/m and there are 3 springs, therefore total stiffness,

$$s = 3 \times 10 \times 10^3 = 30 \times 10^3 \text{ N/m}$$

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{30 \times 10^3}{75}} = 20 \text{ rad/s}$$

## 1. Resistance of the dashpot at unit velocity

Let $c$ = Resistance of the dashpot in newtons at unit velocity *i.e.* in N/m/s,

$x_2$ = Amplitude after one complete oscillation in metres, and

$x_3$ = Amplitude after two complete oscillations in metres.

We know that

$$\frac{x_1}{x_2} = \frac{x_2}{x_3}$$

$\therefore$

$$\left(\frac{x_1}{x_2}\right)^2 = \frac{x_1}{x_3} \quad \ldots \left[\because\ \frac{x_1}{x_3} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} = \frac{x_1}{x_2} \times \frac{x_1}{x_2} = \left(\frac{x_1}{x_2}\right)^2\right]$$

or

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_3}\right)^{1/2} = \left(\frac{0.0384}{0.0064}\right)^{1/2} = 2.45$$

We also know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 2.45 = a \times \frac{2\pi}{\sqrt{(20)^2 - a^2}}$$

$$0.8951 = \frac{a \times 2\pi}{\sqrt{400 - a^2}} \quad \text{or} \quad 0.8 = \frac{a^2 \times 39.5}{400 - a^2} \quad \text{. . . (Squaring both sides)}$$

$\therefore$ $a^2 = 7.94$ or $a = 2.8$

We know that $a = c / 2m$

$\therefore$ $c = a \times 2m = 2.8 \times 2 \times 75 = 420$ N/m/s **Ans.**

**2.** Ratio of the frequency of the damped vibration to the frequency of undamped vibration

Let $f_{n1}$ = Frequency of damped vibration = $\dfrac{\omega_d}{2\pi}$

$f_{n2}$ = Frequency of undamped vibration = $\dfrac{\omega_n}{2\pi}$

$$\therefore \quad \frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \frac{\sqrt{(\omega_n)^2 - a^2}}{\omega_n} = \frac{\sqrt{(20)^2 - (2.8)^2}}{20} = 0.99 \text{ **Ans.**}$$

**3.** Periodic time of damped vibration

We know that periodic time of damped vibration

$$= \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{2\pi}{\sqrt{(20)^2 - (2.8)^2}} = 0.32 \text{ s **Ans.**}$$

**Example 23.16.** *The mass of a single degree damped vibrating system is 7.5 kg and makes 24 free oscillations in 14 seconds when disturbed from its equilibrium position. The amplitude of vibration reduces to 0.25 of its initial value after five oscillations. Determine :* ***1.*** *stiffness of the spring,* ***2.*** *logarithmic decrement, and* ***3.*** *damping factor, i.e. the ratio of the system damping to critical damping.*

**Solution.** Given : $m = 7.5$ kg

Since 24 oscillations are made in 14 seconds, therefore frequency of free vibrations,

$$f_n = 24/14 = 1.7$$

and $\omega_n = 2\pi \times f_n = 2\pi \times 1.7 = 10.7$ rad/s

**1.** Stiffness of the spring

Let $s$ = Stiffness of the spring in N/m.

We know that $(\omega_n)^2 = s / m$ or $s = (\omega_n)^2 m = (10.7)^2\, 7.5 = 860$ N/m **Ans.**

**2.** Logarithmic decrement

Let $x_1$ = Initial amplitude,

$x_6$ = Final amplitude after five oscillations = $0.25\, x_1$ ... (Given)

$$\therefore \quad \frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5 \quad \ldots \left[\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}\right]$$

$\left( x_2 \right)$

$\lfloor \quad x_3 \quad x_4 \quad x_5$

$x_2 \quad x_6 \rfloor$

or $$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\, x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that logarithmic decrement,

$$\delta = \log_e \left(\frac{x_1}{x_2}\right) = \log_e 1.32 = 0.28 \textbf{ Ans.}$$

### 3. Damping factor

Let $c$ = Damping coefficient for the actual system, and

$c_c$ = Damping coefficient for the critical damped system.

We know that logarithmic decrement ( $\delta$ ),

$$0.28 = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{a \times 2\pi}{\sqrt{(10.7)^2 - a^2}}$$

$$0.0784 = \frac{a^2 \times 39.5}{114.5 - a^2} \quad \text{. . . (Squaring both sides)}$$

$8.977 - 0.0784\, a^2 = 39.5\, a^2$ or $a^2 = 0.227$ or $a = 0.476$

We know that $a = c / 2m$ or $c = a \times 2m = 0.476 \times 2 \times 7.5 = 7.2$ N/m/s **Ans.**

and $c_c = 2m.\omega_n = 2 \times 7.5 \times 10.7 = 160.5$ N/m/s **Ans.**

$\therefore$ Damping factor $= c/c_c = 7.2 / 160.5 = 0.045$ **Ans.**

## Fre quency of Under Damped Forced Vibrations

Consider a system consisting of spring, mass and damper as shown in Fig. 23.19. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F \cos \omega.t$$

where $F$ = Static force, and

$\omega$ = Angular velocity of the periodic disturbing force.

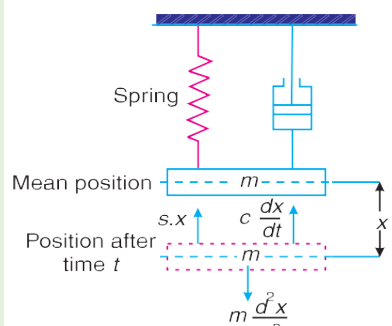



**Fig. 23.19.** Frequency of under damped forced vibrations.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime $t$, the mass is displaced downwards through a distance $x$ from its mean position.

Using the symbols as discussed in the previous article, the equation of motion may be written as

$$m \times \frac{d^2x}{dt^2} = -c \times \frac{dx}{dt} - s.x + F \cos \omega.t$$

or $$m \times \frac{d^2x}{dt^2} + \times \frac{dx}{} + . = \cos \omega.$$

$c$ $s\,x$ $F$ $t$ . . . (*i*)

$dt^2$ $dt$

This equation of motion may be solved either by differential equation method or by graphi- cal method as discussed below :

## 1. Differential equation method

The equation (*i*) is a differential equation of the second degree whose right hand side is some function in *t*. The solution of such type of differential equation consists of two parts ; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where $x_1$ = Complementary function, and

$x_2$ = Particular integral.

The complementary function is same as discussed in the previous article, *i.e.*

$$x_1 = Ce^{-at} \cos (\omega_d t - \theta) \quad \ldots (ii)$$

where $C$ and $\theta$ are constants. Let us now find the value of particular integral as discussed below : Let the particular integral of equation (*i*) is given by

$$x_2 = B_1 \sin \omega.t + B_2 \cos\omega.t \quad \ldots \text{(where } B_1 \text{ and } B_2 \text{ are constants)}$$

$$\therefore \quad \frac{dx}{dt} = B_1.\omega\cos \omega.t - B_2.\omega\sin \omega.t$$

and
$$\frac{d^2x}{dt^2} = -B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos \omega.t$$

Substituting these values in the given differential equation (*i*), we get

$$m(-B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos\omega.t) + c(B_1.\omega\cos \omega.t - B_2.\omega\sin \omega.t) + s(B_1 \sin \omega.t + B_2 \cos\omega.t)$$
$$= F \cos \omega.t$$

or
$$(-m.B_1.\omega^2 - c.\omega.B_2 + s.B_1)\sin \omega.t + (-m.\omega^2.B_2 + c.\omega.B_1 + s.B_2)\cos \omega.t$$
$$= F \cos \omega.t$$

or
$$\left[(s - m.\omega^2)B_1 - c.\omega.B_2\right]\sin\omega.t + \left[c.\omega.B_1 + (s - m.\omega^2)B_2\right]\cos\omega.t$$
$$= F \cos \omega.t + 0 \sin \omega.t$$

Comparing the coefficients of sin $\omega t$ and cos $\omega t$ on the left hand side and right hand side separately, we get

$$(s - m.\omega^2)B_1 - c.\omega.B_2 = 0 \quad \ldots (iii)$$

and
$$c.\omega.B_1 + (s - m.\omega^2)B_2 = F \quad \ldots (iv)$$

Now from equation (*iii*)

$$(s - m.\omega^2)B_1 = c.\omega.B_2$$

$$\therefore \quad B_2 = \frac{s - m.\omega^2}{c.\omega} \times B_1 \quad \ldots (v)$$

Substituting the value of $B_2$ in equation (*iv*)

$$c.\omega B_1 + \frac{(s - m.\omega^2)(s - m.\omega^2)}{c.\omega} \times B_1 = F$$

$$c^2.\omega^2.B_1 + (s - m.\omega^2)^2 B_1 = c.\omega.F$$

$$B_1\left[c^2.\omega^2 + (s - m.\omega^2)^2\right] = c.\omega.F$$

$\therefore$
$$B_1 = \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

and
$$B_2 = \frac{s - m.\omega^2}{c.\omega} \times \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$ . . . [From equation (*v*)]

$$= \frac{F(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

$\therefore$ The particular integral of the differential equation (*i*) is

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t$$

$$= \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \sin \omega.t + \frac{F(s - m.\omega^2)}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos \omega.t$$

$$= \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2}\left[c.\omega \sin \omega.t + (s - m.\omega^2)\cos \omega.t\right]$$ . . . (*vi*)

Let $c.\omega = X \sin \phi$; and $s - m.\omega^2 = X \cos \phi$

$\therefore$ $X = \sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}$ . . . (By squaring and adding)

This machine performs pressing operation, welding operation and material handling.

Note : This picture is given as additional information and is not a direct example of the current chapter.

and $\tan\phi = \dfrac{c.\omega}{s - m.\omega^2}$ or $\phi = \tan^{-1}\left(\dfrac{c.\omega}{s - m.\omega^2}\right)$

Now the equation (*vi*) may be written as

$$x_2 = \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2}\left[X \sin\phi.\sin\omega t + X\cos\phi\cos\omega t\right]$$

$$= \frac{F.X}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega.t - \phi)$$

$$= \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

$\therefore$ The complete solution of the differential equation (*i*) becomes

$$x = x_1 + x_2$$

$$= C.e^{-at}\cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

In actual practice, the value of the complementary function $x_1$ at any time $t$ is much smaller as compared to particular integral $x_2$. Therefore, the displacement $x$, at any time $t$, is given by the particular integral $x_2$ only.

$$\therefore \quad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi) \quad \text{... (vii)}$$

This equation shows that motion is simple harmonic whose circular frequency is $\omega$ and the amplitude is $\dfrac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$.

A little consideration will show that the frequency of forced vibration is equal to the angular velocity of the periodic force and the amplitude of the forced vibration is equal to the maximum displacement of vibration.

$\therefore$ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \quad \text{. . . (viii)}$$

**Notes : 1.** The equations (*vii*) and (*viii*) hold good when steady vibrations of constant amplitude takes place.

**2.** The equation (*viii*) may be written as

$$x_{max} = \frac{F/s}{\sqrt{\dfrac{c^2.\omega^2}{s^2} + \dfrac{(s - m.\omega^2)^2}{s^2}}}$$

. . . (Dividing the numerator and denominator by $s$)

$$= \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{m.\omega^2}{s}\right)^2}} \qquad \ldots \text{(Substituting } F/s = x_o\text{)}$$

where $x_o$ is the deflection of the system under the static force *F*. We know that the natural frequency of free vibrations is given by

$$(\omega_n)^2 = s/m$$

$$\therefore \quad x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (ix)$$

**3.** When damping is negligible, then $c = 0$.

$$\therefore \quad x_{max} = \frac{x_o}{1 - \frac{\omega^2}{(\omega_n)^2}} = \frac{x_o (\omega_n)^2}{(\omega_n)^2 - \omega^2} = \frac{x_o \times s/m}{(\omega_n)^2 - \omega^2} \qquad \ldots \left[\because (\omega_n)^2 = s/m\right]$$

$$\therefore \quad = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} \qquad \ldots (\because F = x_o.s) \ldots (x)$$

**4.** At resonance $\omega = \omega_n$. Therefore the angular speed at which the resonance occurs is

$$\omega = \omega_n = \sqrt{\frac{s}{m}} \text{ rad/s}$$

and

$$x_{max} = x_o \times \frac{s}{c.\omega_n} \qquad \ldots \text{[From equation } (ix)\text{]}$$

## 2. Graphical method

The solution of the equation of motion for a forced and damped vibration may be easily obtained by graphical method as discussed below :

Let us assume that the displacement of the mass (*m*) in the system, as shown in Fig. 23.19, under the action of the applied simple harmonic force $F \cos \omega.t$ is itself simple harmonic, so that it can be represented by the equation,

$$x = A \cos(\omega t - \phi)$$

where *A* is the amplitude of vibration.

Now differentiating the above equation,

$$\frac{dx}{dt} = -\omega.A \sin(\omega.t - \phi) = \omega.A \cos\left[90° + (\omega.t - \phi)\right]$$

and

$$\frac{d^2x}{dt^2} = -\omega^2.A \cos(\omega.t - \phi) = \omega^2.A \cos\left[180° + (\omega.t - \phi)\right]$$

$\therefore$ Elastic force *i.e.* the force required to extend the spring

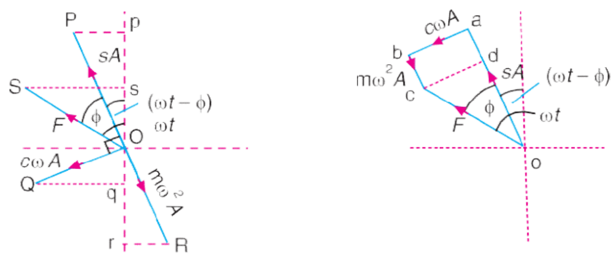
$$= s.x = s.A\cos(\omega.t - \phi)$$

Disturbing force *i.e.* the force required to overcome the resistance of dashpot

$$= c \times \frac{dx}{dt} = c.\omega.A \cos\left[90^\circ + (\omega.t - \phi)\right]$$

and inertia force *i.e.* the force required to accelerate the mass $m$

$$= m \times \frac{d^2x}{dt^2} = m.\omega^2.A \cos\left[180^\circ + (\omega.t - \phi)\right]$$




(a) (*b*)

**Fig. 23.20.** Graphical method.

The algebraic sum of these three forces at any given instant must be equal to the applied force $F\cos\omega t$. These forces are represented graphically in Fig. 23.20 (*a*). The vector *OP* represents, to some suitable scale, the elastic force (of maximum value $s.A$), at an inclination $(\omega.t - \phi)$ to the vertical. The vector *OQ* (of maximum value $c\omega.A$) and vector *OR* (of maximum value $m.\omega^2 A$) represents, to the same scale, the disturbing force and inertia force respectively. The vec- tors *OP*, *OQ* and *OR* are at successive intervals of 90°.

The projected lengths *Op*, *Oq* and *Or* represent the instantaneous values of these forces at time $t$ and *Os* (the algebraic sum of *Op*, *Oq* and *Or*) must represent the value $F\cos\omega.t$ of the applied force at the same instant. Thus the force vector *OS* must be the vector sum of *OP*, *OQ* and *OR* or force *F* must be the vector sum of $s.A$, $c.\omega.A$ and $m.\omega^2.A$, as shown in Fig. 23.20 (*b*). From the geometry of the figure,

$$F = oc = \sqrt{(od)^2 + (cd)^2} = \sqrt{(oa - ad)^2 + (cd)^2}$$

$$= \sqrt{(s.A - m.\omega^2.A)^2 + (c.\omega.A)^2} = A\sqrt{(s - m.\omega^2)^2 + c^2.\omega^2}$$

$$\therefore \quad A\,(\text{or } x_{max}) = \frac{F}{\sqrt{(s - m.\omega^2)^2 + c^2.\omega^2}} \qquad \text{. . . (Same as before)}$$

and

$$\tan\phi = \frac{cd}{od} = \frac{c.\omega.A}{s.A - m.\omega^2.A} = \frac{c.\omega^2}{s - m.\omega^2} \qquad \text{. . . (Same as before)}$$

## Magnific ation Fa ctor or Dynamic Magnifier

It is the ratio of ***maximum displacement of the forced vibration*** ($x_{max}$) ***to the deflection due to the static force*** $F(x_o)$. We have proved in the previous article that the maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

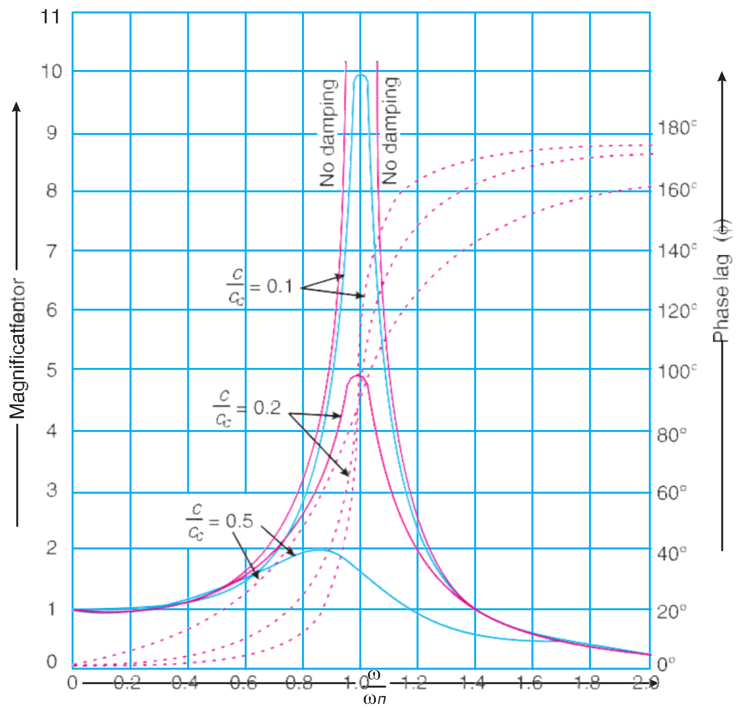



**Fig. 23.21.** Relationship between magnification factor and phase angle for different values of $\omega/\omega_n$.

∴ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \quad \text{... } (i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\left[\because \frac{c.\omega}{s} = \frac{2c.\omega}{2m \times \frac{s}{m}} = \frac{2c.\omega}{2m(\omega_n)^2} = \frac{2c.\omega}{c_c.\omega_n}\right]$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force $F$ (*i.e.* $x_o$) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e.* $x_{max}$) by the harmonic force $F \cos \omega.t$

$$\therefore \qquad x_{max} = x_o \times D$$

Fig. 23.21 shows the relationship between the magnification factor ($D$) and phase angle $\phi$ for different value of $\omega / \omega_n$ and for values of damping factor $c/c_c$ = 0.1, 0.2 and 0.5.

**Notes: 1.** If there is no damping (*i.e.* if the vibration is undamped), then $c = 0$. In that case, magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} = \frac{(\omega_n)^2}{(\omega_n)^2 - \omega^2}$$

**2.** At resonance, $\omega = \omega_n$ . Therefore magnification factor,

$$D = \frac{x_{max}}{x_o} = \frac{s}{c.\omega_n}$$

Depending upon the case bridges can be treated as beams subjected to uniformly distributed leads and point loads.

**Example 23.17.** *A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1.5 kN per metre per second.*

*Considering that the steady state of vibration is reached ; determine : **1.** the amplitude of forced vibrations, when the driving shaft of the engine rotates at 480 r.p.m., and **2.** the speed of the driving shaft at which resonance will occur.*

**Solution :** Given. $m$ = 300 kg; $\delta$ = 2 mm = $2 \times 10^{-3}$ m ; $m_1$ = 20 kg ; $l$ = 150 mm = 0.15 m ; $c$ = 1.5 kN/m/s = 1500 N/m/s ; $N$ = 480 r.p.m. or $\omega = 2\pi \times 480 / 60$ = 50.3 rad/s

## 1. Amplitude of the forced vibrations

We know that stiffness of the frame,

$$s = m.g / \delta = 300 \times 9.81/2 \times 10^{-3} = 1.47 \times 10^{6} \text{ N/m}$$

Since the length of stroke ( $l$ ) = 150 mm = 0.15 m, therefore radius of crank,

$$r = l / 2 = 0.15 / 2 = 0.075 \text{ m}$$

We know that the centrifugal force due to the reciprocating parts or the static force,

$$F = m_1.\omega^2.r = 20\,(50.3)^2\, 0.075 = 3795 \text{ N}$$

∴ Amplitude of the forced vibration (maximum),

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{3795}{\sqrt{(1500)^2\,(50.3)^2 + [1.47 \times 10^6 - 300\,(50.3)^2\,]^2}}$$

$$= \frac{3795}{\sqrt{5.7 \times 10^9 + 500 \times 10^9}} = \frac{3795}{710 \times 10^3} = 5.3 \times 10^{-3} \text{ m}$$

= 5.3 mm **Ans.**

## 2. Speed of the driving shaft at which the resonance occurs

Let $N$ = Speed of the driving shaft at which the resonance occurs in r.p.m.

We know that the angular speed at which the resonance occurs,

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.47 \times 10^6}{300}} = 70 \text{ rad/s}$$

∴ $N = \omega \times 60 / 2\pi = 70 \times 60 / 2\pi$ = 668.4 r.p.m. **Ans.**

**Example 23.18.** *A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the vertical direction, find the amplitude of the forced vibrations. What is its value of resonance ?*

**Solution.** Given : $m$ = 10 kg ; $s$ = 10 N/mm = 10 × 10³ N/m ; $x_5 = \dfrac{x_1}{10}$

Since the periodic force, $F_x = F \cos \omega.t$ = 150 cos 50 $t$ , therefore

Static force, $F$ = 150 N

and angular velocity of the periodic disturbing force,

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10 \times 10^3}{10}} = 31.6 \text{ rad/s}$$

## Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude ($x_1$) to the final amplitude after four complete oscillations ($x_5$) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \qquad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \quad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \qquad \ldots \left(\because x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e \left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 1.78 = a \times \frac{2\pi}{\sqrt{(31.6)^2 - a^2}} \quad \text{or} \quad 0.576 = \frac{a \times 2\pi}{\sqrt{1000 - a^2}}$$

Squaring both sides and rearranging,

$$39.832\, a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s and deflection of the system produced by the static force $F$,

$$x_o = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

We know that amplitude of the forced vibrations,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2 \omega^2}{s^2} + \left[1 - \frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2 (50)^2}{(10 \times 10^3)^2} + \left[1 - \left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083 + 2.25}}$$

$$= \frac{0.015}{1.53} = 9.8 \times 10^{-3} \text{ m} = 9.8 \text{ mm} \quad \textbf{Ans.}$$

## Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015 \times \frac{10 \times 10^3}{57.54 \times 31.6} = 0.0822 \text{ m} = 82.2 \text{ mm} \quad \textbf{Ans.}$$

**Example 23.19.** *A body of mass 20 kg is suspended from a spring which deflects 15 mm under this load. Calculate the frequency of free vibrations and verify that a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just-sufficient to make the motion aperiodic.*

*If when damped to this extent, the body is subjected to a disturbing force with a maximum value of 125 N making 8 cycles/s, find the amplitude of the ultimate motion.*

**Solution .** Given : $m$ = 20 kg ; $\delta$ = 15 mm = 0.015 m ; $c$ = 1000 N/m/s ; $F$ = 125 N ; $f$ = 8 cycles/s

## Frequency of free vibrations

We know that frequency of free vibrations,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.015}} = 4.07 \text{ Hz } \textbf{Ans.}$$

The critical damping to make the motion aperiodic is such that damped frequency is zero, *i.e.*

$$\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$$

$$\therefore \quad c = \sqrt{\frac{s}{m}\times 4m^2} = \sqrt{4\,s.m} = \sqrt{4\times\frac{m.g}{\delta}\times m} \qquad \ldots\left(\because s = \frac{m.g}{\delta}\right)$$

$$= \sqrt{4\times\frac{20\times 9.81}{0.015}\times 20} = 1023 \text{ N/m/s}$$

This means that the viscous damping force is 1023 N at a speed of 1 m/s. Therefore a viscous damping force amounting to approximately 1000 N at a speed of 1 m/s is just sufficient to make the motion aperiodic. **Ans.**

## Amplitude of ultimate motion

We know that angular speed of forced vibration,

$$\omega = 2\pi\times f = 2\pi\times 8 = 50.3 \text{ rad/s}$$

and stiffness of the spring, $s = m.g/\delta = 20 \times 9.81 / 0.015 = 13.1 \times 10^3$ N/m

$\therefore$ Amplitude of ultimate motion *i.e.* maximum amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{125}{\sqrt{(1023)^2\,(50.3)^2 + [13.1\times 10^3 - 20\,(50.3)^2]^2}}$$

$$= \frac{125}{\sqrt{2600\times 10^6 + 1406\times 10^6}} = \frac{125}{63.7\times 10^3} = 1.96\times 10^{-3} \text{ m}$$

= 1.96 mm **Ans.**

**Example 23.20.** *A machine part of mass 2 kg vibrates in a viscous medium. Determine the damping coefficient when a harmonic exciting force of 25 N results in a resonant amplitude of 12.5 mm with a period of 0.2 second. If the system is excited by a harmonic force of frequency 4 Hz what will be the percentage increase in the amplitude of vibration when damper is removed as compared with that with damping.*

**Solution .** Given : $m$ = 2 kg ; $F$ = 25 N ; Resonant $x_{max}$ = 12.5 mm = 0.0125 m ; $t_p$ = 0.2 s ; $f$ = 4 Hz

## Damping coefficient

Let $c$ = Damping coefficient in N/m/s. We know that natural circular frequency of the exicting force,

$$\omega_n = 2\pi / t_p = 2\pi / 0.2 = 31.42 \text{ rad/s}$$

We also know that the maximum amplitude of vibration at resonance ($x_{max}$),

$$0.0125 = \frac{F}{c.\omega_n} = \frac{25}{c \times 31.42} = \frac{0.796}{c} \quad \text{or } c = 63.7 \text{ N/m/s } \textbf{Ans.}$$

## Percentage increase in amplitude

Since the system is excited by a harmonic force of frequency ($f$) = 4 Hz, therefore corresponding circular frequency

$$\omega = 2\pi \times f = 2\pi \times 4 = 25.14 \quad \text{rad/s}$$

We know that maximum amplitude of vibration with damping,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{25}{\sqrt{(63.7)^2 (25.14)^2 + [2\,(31.42)^2 - 2\,(25.14)^2]^2}}$$

$$\ldots \left[\because (\omega_n)^2 = s/m \text{ or } s = m(\omega_n)^2\right]$$

$$= \frac{25}{\sqrt{2.56 \times 10^6 + 0.5 \times 10^6}} = \frac{25}{1749} = 0.0143 \text{ m} = 14.3 \text{ mm}$$

and the maximum amplitude of vibration when damper is removed,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]} = \frac{25}{2[(31.42)^2 - (25.14)^2]} = \frac{25}{710} = 0.0352 \text{ m}$$

$$= 35.2 \text{ mm}$$

∴ Percentage increase in amplitude

$$= \frac{35.2 - 14.3}{14.3} = 1.46 \quad \text{or} \quad 146\% \ \textbf{Ans.}$$

**Example 23.21.** *The time of free vibration of a mass hung from the end of a helical spring is 0.8 second. When the mass is stationary, the upper end is made to move upwards with a displacement y metre such that y = 0.018 sin 2 π t, where t is the time in seconds measured from the beginning of the motion. Neglecting the mass of the spring and any damping effects, determine the vertical distance through which the mass is moved in the first 0.3 second.*

**Solution.** Given : $t_p = 0.8$ s ; $y = 0.018 \sin 2\pi t$

Let $m$ = Mass hung to the spring in kg, and

$s$ = Stiffness of the spring in N/m.

We know that time period of free vibrations ($t_p$),

$$0.8 = 2\pi\sqrt{\frac{m}{s}} \quad \text{or} \quad \frac{m}{s} = \left(\frac{0.8}{2\pi}\right)^2 = 0.0162$$

If $x$ metres is the upward displacement of mass $m$ from its equilibrium position after time $t$ seconds, the equation of motion is given by

$$m \times \frac{d^2x}{dt^2} = s(y - x) \quad \text{or} \quad \frac{m}{s} \times \frac{d^2x}{dt^2} + x = y = 0.018 \sin 2\pi t$$

The solution of this differential equation is

$$x = A \sin \sqrt{\frac{s}{m}} \times t + B \cos \sqrt{\frac{s}{m}} \times t + \frac{0.018 \sin 2\pi t}{1 - \left(\frac{2\pi}{\sqrt{s/m}}\right)^2}$$

. . . (where $A$ and $B$ are constants)

$$= A \sin \frac{t}{\sqrt{0.0162}} + B \cos \frac{t}{\sqrt{0.0162}} + \frac{0.018 \sin 2\pi t}{1 - 4\pi^2 \times 0.0162}$$

$$= A \sin 7.85t + B \cos 7.85t + 0.05 \sin 2\pi t \qquad \text{. . . } (i)$$

Now when $t = 0$, $x = 0$, then from equation **(*i*)**, $B = 0$.

Again when $t = 0$, $dx/dt = 0$.

Therefore differentiating equation **(*i*)** and equating to zero, we have

$$dx/dt = 7.85A \cos 7.85t + 0.05 \times 2\pi \cos 2\pi t = 0 \qquad \text{. . . } (\because B = 0)$$

or $$7.85 A \cos 7.85t = -0.05 \times 2\pi \cos 2\pi t$$

$\therefore$ $$A = -0.05 \times 2\pi / 7.85 = -0.04 \qquad \text{. . . } (\because t = 0)$$

Now the equation **(*i*)** becomes

$$x = -0.04 \sin 7.85t + 0.05 \sin 2\pi t \qquad \text{. . . } (\because B = 0) \text{ . . . } (ii)$$

$\therefore$ Vertical distance through which the mass is moved in the first 0.3 second (*i.e.* when $t$ = 0.3 s),

$$= -0.04 \sin (7.85 \times 0.3) + 0.05 \sin (2\pi \times 0.3)$$

. . . [ Substituting $t = 0.3$ in equation **(*ii*)**]

$$= -0.04 \times 0.708 + 0.05 \times 0.951 = -0.0283 + 0.0476 = 0.0193 \text{ m}$$

$= 19.3$ mm **Ans.**

## Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimise the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 23.22. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

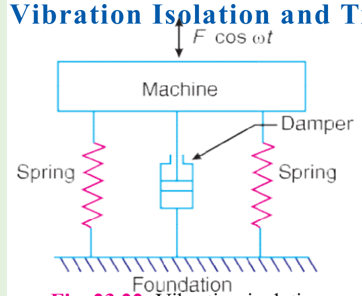



**Fig. 23.22.** Vibration isolation.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass $m$ supported by a spring of stiffness $s$, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.

The ratio of the force transmitted ($F_T$) to the force applied ($F$) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the fol- lowing two forces :

**1.** Spring force or elastic force which is equal to $s.\, x_{max}$, and

**2.** Damping force which is equal to $c.\,\omega\,.x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the force transmitted,

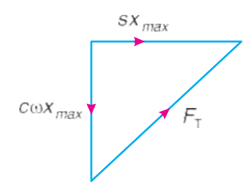



**Fig. 23.23**

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

$\therefore$ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

We know that

$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because\ x_o = \frac{F}{s}\right)$$

$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)^2} \qquad \ldots \left(\because\ \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)$$

We have seen in Art. 23.17 that the magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\therefore \qquad \varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (i)$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2} \qquad \ldots (ii)$$

From above, we see that when $\omega/\omega_n > 1$, $\varepsilon$ is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force $(F \cos\omega.t)$. The value of $\omega/\omega_n$ must be greater than $\sqrt{2}$ if $\varepsilon$ is to be less than 1 and it is the numerical value of $\varepsilon$, independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation **(*ii*)** in the following form, *i.e.*

$$\varepsilon = \frac{1}{(\omega/\omega_n)^2 - 1} \qquad \ldots (iii)$$

Fig. 23.24 is the graph for different values of damping factor $c/c_c$ to show the variation of transmissibility ratio ( $\varepsilon$ ) against the ratio $\omega/\omega_n$.

**1.** When $\omega/\omega_n = \sqrt{2}$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor $c/c_c$.

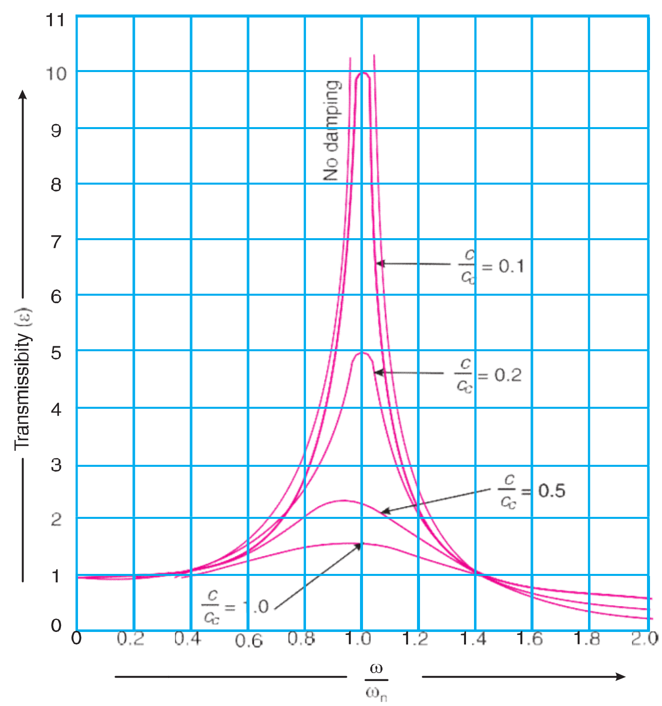



**Fig. 23.24.** Graph showing the variation of transmissibility ratio.

**2.** When $\omega/\omega_n < \sqrt{2}$, then $\varepsilon > 1$ for all values of damping factor $c/c_c$. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

**3.** When $\omega/\omega_n > \sqrt{2}$, then $\varepsilon < 1$ for all values of damping factor $c/c_c$. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega_n > \sqrt{2}$.

We also see from the curves in Fig. 23.24 that the damping is detrimental beyond $\omega/\omega_n > \sqrt{2}$ and advantageous only in the region $\omega/\omega_n < \sqrt{2}$. It is thus concluded that for the vibration isolation, dampers need not to be provided but in order to limit resonance amplitude, stops may be provided.

**Example 23.22.** *The mass of an electric motor is 120 kg and it runs at 1500 r.p.m. The armature mass is 35 kg and its C.G. lies 0.5 mm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the mass of the motor is equally distributed among the five springs.*

*Determine : **1.** stiffness of each spring; **2.** dynamic force transmitted to the base at the operating speed; and **3.** natural frequency of the system.*

**Solution.** Given $m_1 = 120$ kg ; $m_2 = 35$ kg; $r = 0.5$ mm $= 5 \times 10^{-4}$ m; $\varepsilon = 1/11$; $N = 1500$ r.p.m. or $\omega = 2\pi \times 1500/60 = 157.1$ rad/s ;

**1. Stiffness of each spring**

Let $s$ = Combined stiffness of the spring in N-m, and

$\omega_n$ = Natural circular frequency of vibration of the machine in rad/s.

We know that transmissibility ratio ($\varepsilon$),

$$\frac{1}{11} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(157.1)^2 - (\omega_n)^2}$$

or $$(157.1)^2 - (\omega_n)^2 = 11(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 2057 \quad \text{or} \quad \omega_n = 45.35 \text{ rad/s}$$

We know that $\omega_n = \sqrt{s/m_1}$

$$s = m_1(\omega_n)^2 = 120 \times 2057 = 246\,840 \text{ N/m}$$

Since these are five springs, therefore stiffness of each spring

$= 246\,840 / 5 = 49\,368$ N/m **Ans.**

**2. Dynamic force transmitted to the base at the operating speed (i.e. 1500 r.p.m. or 157.1 rad/s)**

We know that maximum unbalanced force on the motor due to armature mass,

$$F = m_2\,\omega^2 \cdot r = 35\,(157.1)^2\, 5 \times 10^{-4} = 432 \text{ N}$$

$\therefore$ Dynamic force transmitted to the base,

$$F_T = \varepsilon . F = \frac{1}{11} \times 432 = 39.27 \text{ N}$$ **Ans.**

**3. Natural frequency of the system**

We have calculated above that the natural frequency of the system,

$\omega_n = 45.35$ rad/s **Ans.**

**Example 23.23.** *A machine has a mass of 100 kg and unbalanced reciprocating parts of mass 2 kg which move through a vertical stroke of 80 mm with simple harmonic motion. The machine is mounted on four springs, symmetrically arranged with respect to centre of mass, in such a way that the machine has one degree of freedom and can undergo vertical displacements only.*

*Neglecting damping, calculate the combined stiffness of the spring in order that the force transmitted to the foundation is 1 / 25 th of the applied force, when the speed of rotation of machine crank shaft is 1000 r.p.m.*

*When the machine is actually supported on the springs, it is found that the damping reduces the amplitude of successive free vibrations by 25%. Find :* ***1.*** *the force transmitted to foundation at 1000 r.p.m.,* ***2.*** *the force transmitted to the foundation at resonance, and* ***3.*** *the amplitude of the forced vibration of the machine at resonance.*

**Solution.** Given : $m_1 = 100$ kg ; $m_2 = 2$ kg ; $l = 80$ mm $= 0.08$ m ; $\varepsilon = 1/25$ ; $N = 1000$ r.p.m. or $\omega = 2\pi \times 1000 / 60 = 104.7$ rad/s

## Combined stiffness of springs

Let $s$ = Combined stiffness of springs in N/m, and

$\omega_n$ = Natural circular frequency of vibration of the machine in rad/s.

We know that transmissibility ratio ( $\varepsilon$ ),

$$\frac{1}{25} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(104.7)^2 - (\omega_n)^2}$$

or $$(104.7)^2 - (\omega_n)^2 = 25(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 421.6 \quad \text{or} \quad \omega_n = 20.5 \text{ rad/s}$$

We know that $$\omega_n = \sqrt{s/m_1}$$

$$\therefore \quad s = m_1(\omega_n)^2 = 100 \times 421.6 = 42\ 160 \text{ N/m}$$ **Ans.**

## 1. Force transmitted to the foundation at 1000 r.p.m.

Let $F_T$ = Force transmitted, and

$x_1$ = Initial amplitude of vibration.

Since the damping reduces the amplitude of successive free vibrations by 25%, therefore final amplitude of vibration,

$$x_2 = 0.75\,x_1$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} \quad \text{or} \quad \log_e\left(\frac{x_1}{0.75x_1}\right) = \frac{a \times 2\pi}{\sqrt{421.6 - a^2}}$$

Squaring both sides,

$$(0.2877)^2 = \frac{a^2 \times 4\pi^2}{421.6 - a^2} \quad \text{or} \quad 0.083 = \frac{39.5\,a^2}{421.6 - a^2}$$

$$\ldots\left[\because \log_e\left(\frac{1}{0.75}\right) = \log_e 1.333 = 0.2877\right]$$

$$35 - 0.083\,a^2 = 39.5\,a^2 \quad \text{or} \quad a^2 = 0.884 \quad \text{or} \quad a = 0.94$$

We know that damping coefficient or damping force per unit velocity,

$$c = a \times 2m_1 = 0.94 \times 2 \times 100 = 188 \text{ N/m/s}$$

and critical damping coefficient,

$$c_c = 2m.\omega_n = 2 \times 100 \times 20.5 = 4100 \text{ N/m/s}$$

$\therefore$ Actual value of transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$= \frac{\sqrt{1 + \left(\frac{2 \times 188 \times 104.7}{4100 \times 20.5}\right)^2}}{\sqrt{\left(\frac{2 \times 188 \times 104.7}{4100 \times 20.5}\right)^2 + \left[1 - \left(\frac{104.7}{20.5}\right)^2\right]^2}} = \frac{\sqrt{1 + 0.22}}{\sqrt{0.22 + 629}}$$

$$= \frac{1.104}{25.08} = 0.044$$

We know that the maximum unbalanced force on the machine due to reciprocating parts,

$$F = m_2.\omega^2.r = 2(104.7)^2 (0.08/2) = 877 \text{ N} \quad \ldots (\because r = l/2)$$

$\therefore$ Force transmitted to the foundation,

$$F_T = \varepsilon.F = 0.044 \times 877 = 38.6 \text{ N} \quad \textbf{Ans.} \ldots\ldots\ldots (\because \varepsilon = F_T/F)$$

## 2. Force transmitted to the foundation at resonance

Since at resonance, $\omega = \omega_n$, therefore transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1 + \left(\frac{2c}{c_c}\right)^2}}{\sqrt{\left(\frac{2c}{c_c}\right)^2}} = \frac{\sqrt{1 + \left(\frac{2 \times 188}{4100}\right)^2}}{\sqrt{\left(\frac{2 \times 188}{4100}\right)^2}} = \frac{\sqrt{1 + 0.0084}}{0.092} = 10.92$$

and maximum unbalanced force on the machine due to reciprocating parts at resonance speed $\omega_n$,

$$F = m_2 (\omega_n)^2 r = 2(20.5)^2 (0.08/2) = 33.6 \text{ N} \quad \ldots (\because r = l/2)$$

$\therefore$ Force transmitted to the foundation at resonance,

$$F_T = \varepsilon.F = 10.92 \times 33.6 = 367 \text{ N} \quad \textbf{Ans.}$$

## 3. Amplitude of the forced vibration of the machine at resonance

We know that amplitude of the forced vibration at resonance

$$= \frac{\text{Force transmitted at resonance}}{\text{Combined stiffness}} = \frac{367}{42\,160} = 8.7 \times 10^{-3} \text{ m}$$

$= 8.7$ mm **Ans.**

**Example 23.24.** *A single-cylinder engine of total mass 200 kg is to be mounted on an elastic support which permits vibratory movement in vertical direction only. The mass of the piston is 3.5 kg and has a vertical reciprocating motion which may be assumed simple harmonic with a stroke of 150 mm. It is desired that the maximum vibratory force transmitted through the elastic support to the foundation shall be 600 N when the engine speed is 800 r.p.m. and less than this at all higher speeds.*

*1. Find the necessary stiffness of the elastic support, and the amplitude of vibration at 800 r.p.m., and*

*2. If the engine speed is reduced below 800 r.p.m. at what speed will the transmitted force again becomes 600 N?*

**Solution.** Given : $m_1 = 200$ kg ; $m_2 = 3.5$ kg ; $l = 150$ mm = 0.15 mm or $r = l/2 = 0.075$ m; $F_T = 600$ N ; $N = 800$ r.p.m. or $\omega = 2\pi \times 800 / 60 = 83.8$ rad/s

We know that the disturbing force at 800 r.p.m.,

$F$ = Centrifugal force on the piston

$$= m_2 . \omega^2 . r = 3.5\,(83.8)^2\,0.075 = 1843 \text{ N}$$

## 1. Stiffness of elastic support and amplitude of vibration

Let $s$ = Stiffness of elastic support in N/m, and

$x_{max}$ = Max. amplitude of vibration in metres.

Since the max. vibratory force transmitted to the foundation is equal to the force on the elastic support (neglecting damping), therefore

Max. vibratory force transmitted to the foundation,

$F_T$ = Force on the elastic support

= Stiffness of elastic support × Max. amplitude of vibration

$$= s \times x_{max} = s \times \frac{F}{m\left[\omega^2 - (\omega_n)^2\right]}$$

$$= s \times \frac{F}{m\left(\omega^2 - \frac{s}{m}\right)} = \frac{F.s}{m.\omega^2 - s} \qquad \ldots \left[\because (\omega_n)^2 = \frac{s}{m}\right]$$

$$\therefore \qquad 600 = \frac{1843 \times s}{200\,(83.8)^2 - s} = \frac{1843\, s}{1.4 \times 10^6 - s} \qquad \ldots \text{(Substituting } m = m_1)$$

---

* The equation **(*x*)** of Art. 23.16 is

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - \omega^2\right]}$$

Since the max. vibratory force transmitted to the foundation through the elastic support decreases at all higher speeds (*i.e.* above $N = 800$ r.p.m. or $\omega = 83.8$ rad/s), therefore we shall use

$$x_{max} = \frac{F}{m\left[\omega^2 - (\omega_n)^2\right]}$$

or $840 \times 10^6 - 600\, s = 1843\, s$

$\therefore \quad s = 0.344 \times 10^6 = 344 \times 10^3$ N/m **Ans.**

and maximum amplitude of vibration,

$$x_{max} = \frac{F}{m.\omega^2 - s} = \frac{1843}{200(83.8)^2 - 344 \times 10^3} = \frac{1843}{1056 \times 10^3} \text{ m}$$

$$= 1.745 \times 10^{-3} \text{ m} = 1.745 \text{ mm } \textbf{Ans.}$$

## 2. Speed at the which the transmitted force again becomes 600 N

The transmitted force will rise as the speed of the engine falls and passes through reso- nance. There will be a speed below resonance at which the transmitted force will again equal to 600 N. Let this speed be $\omega_1$ rad/s (or $N_1$ r.p.m.).

$\therefore$ Disturbing force, $F = m_2 (\omega_1)^2 r = 3.5 (\omega_1)^2\, 0.075 = 0.2625 (\omega_1)^2$ N

Since the engine speed is reduced below $N_1$ = 800 r.p.m., therefore in this case, max, amplitude of vibration,

$$x_{max} = \frac{F}{m\left[(\omega_n)^2 - (\omega_1)^2\right]} = \frac{F}{m\left[\frac{s}{m} - (\omega_1)^2\right]} = \frac{F}{s - m(\omega_1)^2}$$

and Force transmitted $= s \times \dfrac{F}{s - m(\omega_1)^2}$

$$\therefore \quad 600 = 344 \times 10^3 \times \frac{0.2625(\omega_1)^2}{344 \times 10^3 - 200(\omega_1)^2} = \frac{90.3 \times 10^3 (\omega_1)^2}{344 \times 10^2 - 200(\omega_1)^2}$$

. . . (Substituting $m = m_1$)

$$206.4 \times 10^6 - 120 \times 10^3 (\omega_1)^2 = 90.3 \times 10^3 (\omega_1)^2 \quad \text{or} \quad (\omega_1)^2 = 981$$

$\therefore \quad \omega_1 = 31.32$ rad/s or $N_1 = 31.32 \times 60 / 2\pi = 299$ r.p.m. **Ans.**

# EXERCISES

1. A shaft of 100 mm diameter and 1 metre long is fixed at one end and other end carries a flywheel of mass 1 tonne. Taking Young's modulus for the shaft material as 200 GN/m$^2$, find the natural frequency of longitudinal and transverse vibrations. **[Ans. 200 Hz ; 8.6 Hz]**
2. A beam of length 10 m carries two loads of mass 200 kg at distances of 3 m from each end together with a central load of mass 1000 kg. Calculate the frequency of transverse vibrations. Neglect the mass of the beam and take $I = 10^9$ mm$^4$ and $E = 205 \times 10^3$ N/mm$^2$. **[Ans. 13.8 Hz]**
3. A steel bar 25 mm wide and 50 mm deep is freely supported at two points 1 m apart and carries a mass of 200 kg in the middle of the bar. Neglecting the mass of the bar, find the frequency of transverse vibration.

   If an additional mass of 200 kg is distributed uniformly over the length of the shaft, what will be the frequency of vibration ? Take $E$ = 200 GN/m$^2$. **[Ans. 17.8 Hz ; 14.6 Hz]**
4. A shaft 1.5 m long is supported in flexible bearings at the ends and carries two wheels each of 50 kg mass. One wheel is situated at the centre of the shaft and the other at a distance of 0.4 m from the centre towards right. The shaft is hollow of external diameter 75 mm and inner diameter 37.5 mm. The density of the shaft material is 8000 kg/m$^3$. The Young's modulus for the shaft material is 200 GN/m$^2$. Find the frequency of transverse vibration. **[Ans. 33.2 Hz]**

5. A shaft of diameter 10 mm carries at its centre a mass of 12 kg. It is supported by two short bearings, the centre distance of which is 400 mm. Find the whirling speed : 1. neglecting the mass of the shaft, and 2. taking the mass of the shaft also into consideration. The density of shaft material is 7500 kg/m$^3$. **[Ans. 748 r.p.m.; 744 r.p.m.]**
6. A shaft 180 mm diameter is supported in two bearings 2.5 metres apart. It carries three discs of mass 250 kg, 500 kg and 200 kg at 0.6 m, 1.5 m and 2 m from the left hand. Assuming the mass of the shaft 190 kg/m, determine the critical speed of the shaft. Young's modulus for the material of the shaft is 211 GN/m$^2$. **[Ans. 18.8 r.p.m.]**
7. A shaft 12.5 mm diameter rotates in long bearings and a disc of mass 16 kg is secured to a shaft at the middle of its length. The span of the shaft between the bearing is 0.5 m. The mass centre of the disc is 0.5 mm from the axis of the shaft. Neglecting the mass of the shaft and taking $E$ = 200 GN/m$^2$, find : 1 critical speed of rotation in r.p.m., and 2. the range of speed over which the stress in the shaft due to bending will not exceed 120 MN/m$^2$. Take the static deflection of the shaft for a beam fixed at both ends, *i.e.* $\delta = \dfrac{Wl^3}{192EI}$. **[Ans. 1450 r.p.m. ; 1184 to 2050 r.p.m.]**
8. A vertical shaft 25 mm diameter and 0.75 m long is mounted in long bearings and carries a pulley of mass 10 kg midway between the bearings. The centre of pulley is 0.5 mm from the axis of the shaft. Find (*a*) the whirling speed, and (*b*) the bending stress in the shaft, when it is rotating at 1700 r.p.m. Neglect the mass of the shaft and $E$ = 200 GN/m$^2$. **[Ans. 3996 r.p.m ; 12.1 MN/m$^2$]**
9. A shaft 12 mm in diameter and 600 mm long between long bearings carries a central mass of 4 kg. If the centre of gravity of the mass is 0.2 mm from the axis of the shaft, compute the maximum flexural stress in the shaft when it is running at 90 per cent of its critical speed. The value of Young's modulus of the material of the shaft is 200 GN/m$^2$. **[Ans. 14.8 kN/m$^2$]**
10. A vibrating system consists of a mass of 8 kg, spring of stiffness 5.6 N/mm and a dashpot of damping coefficient of 40 N/m/s. Find (*a*) damping factor, (*b*) logarithmic decrement, and (*c*) ratio of the two consecutive amplitudes. **[Ans. 0.094 ; 0.6 ; 1.82]**
11. A body of mass of 50 kg is supported by an elastic structure of stiffness 10 kN/m. The motion of the body is controlled by a dashpot such that the amplitude of vibration decreases to one-tenth of its original value after two complete vibrations. Determine : 1. the damping force at 1 m/s ; 2. the damping ratio, and 3. the natural frequency of vibration. **[Ans. 252 N/m/s ; 0.178 ; 2.214 Hz]**
12. A mass of 85 kg is supported on springs which deflect 18 mm under the weight of the mass. The vibrations of the mass are constrained to be linear and vertical and are damped by a dashpot which reduces the amplitude to one quarter of its initial value in two complete oscillations. Find : 1. the magnitude of the damping force at unit speed, and 2. the periodic time of damped vibration. **[Ans. 435 N/m/s ; 0.27 s]**
13. The mass of a machine is 100 kg. Its vibrations are damped by a viscous dash pot which diminishes amplitude of vibrations from 40 mm to 10 mm in three complete oscillations. If the machine is mounted on four springs each of stiffness 25 kN/m, find (*a*) the resistance of the dash pot at unit velocity, and (*b*) the periodic time of the damped vibration. **[Ans. 6.92 N/m/s ; 0.2 s]**
14. A mass of 7.5 kg hangs from a spring and makes damped oscillations. The time for 60 oscillations is 35 seconds and the ratio of the first and seventh displacement is 2.5. Find (*a*) the stiffness of the spring, and (*b*) the damping resistance in N/m/s. If the oscillations are critically damped, what is the damping resistance required in N/m/s ? **[Ans. 870 N/m ; 3.9 N/m/s ; 162 N/m/s]**
15. A mass of 5 kg is supported by a spring of stiffness 5 kN/m. In addition, the motion of mass is controlled by a damper whose resistance is proportional to velocity. The amplitude of vibration reduces to 1/15th of the initial amplitude in four complete cycles. Determine the damping force per unit velocity and the ratio of the frequencies of the damped and undamped vibrations. **[Ans. 34 N/m/s : 0.994]**
16. A mass of 50 kg suspended from a spring produces a statical deflection of 17 mm and when in motion it experiences a viscous damping force of value 250 N at a velocity of 0.3 m/s. Calculate the periodic time of damped vibration. If the mass is then subjected to a periodic disturbing force having a maximum value of 200 N and making 2 cycles/s, find the amplitude of ultimate motion. **[Ans. 0.262 s ; 8.53 mm]**

17. A mass of 50 kg is supported by an elastic structure of total stiffness 20 kN/m. The damping ratio of the system is 0.2. A simple harmonic disturbing force acts on the mass and at any time $t$ seconds, the force is 60 cos 10 $t$ newtons. Find the amplitude of the vibrations and the phase angle caused by the damping. **[Ans. 3.865 mm ; 14.93°]**
18. A machine of mass 100 kg is supported on openings of total stiffness 800 kN/m and has a rotating unbalanced element which results in a disturbing force of 400 N at a speed of 3000 r.p.m. Assuming the damping ratio as 0.25, determine : 1. the amplitude of vibrations due to unbalance ; and 2. the transmitted force. **[Ans. 0.04 mm ; 35.2 N]**
19. A mass of 500 kg is mounted on supports having a total stiffness of 100 kN/m and which provides viscous damping, the damping ratio being 0.4. The mass is constrained to move vertically and is subjected to a vertical disturbing force of the type $F \cos \omega t$. Determine the frequency at which resonance will occur and the maximum allowable value of $F$ if the amplitude at resonance is to be restricted to 5 mm. **[Ans. 2.25 Hz ; 400 N]**
20. A machine of mass 75 kg is mounted on springs of stiffness 1200 kN/m and with an assumed damping factor of 0.2. A piston within the machine of mass 2 kg has a reciprocating motion with a stroke of 80 mm and a speed of 3000 cycles/min. Assuming the motion to be simple harmonic, find : 1. the amplitude of motion of the machine, 2. its phase angle with respect to the exciting force, 3. the force transmitted to the foundation, and 4. the phase angle of transmitted force with respect to the exciting force. **[Ans. 1.254 mm ; 169.05° ; 2132 N ; 44.8°]**

## DO YOU KNOW?

1. What are the causes and effects of vibrations ?
2. Define, in short, free vibrations, forced vibrations and damped vibrations.
3. Discuss briefly with neat sketches the longitudinal, transverse and torsional free vibrations.
4. Derive an expression for the natural frequency of free transverse and longitudinal vibrations by equilibrium method.
5. Discuss the effect of inertia of the shaft in longitudinal and transverse vibrations.
6. Deduce an expression for the natural frequency of free transverse vibrations for a simply supported shaft carrying uniformly distributed mass of $m$ kg per unit length.
7. Deduce an expression for the natural frequency of free transverse vibrations for a beam fixed at both ends and carrying a uniformly distributed mass of $m$ kg per unit length.
8. Establish an expression for the natural frequency of free transverse vibrations for a simply supported beam carrying a number of point loads, by ($a$) Energy method ; and ($b$) Dunkerley's method.
9. Explain the term 'whirling speed' or 'critical speed' of a shaft. Prove that the whirling speed for a rotating shaft is the same as the frequency of natural transverse vibration.
10. Derive the differential equation characterising the motion of an oscillation system subject to viscous damping and no periodic external force. Assuming the solution to the equation, find the frequency of oscillation of the system.
11. Explain the terms 'under damping, critical damping' and 'over damping'
12. A thin plate of area $A$ and mass $m$ is attached to the end of a spring and is allowed to oscillate in a viscous fluid, as shown in Fig. 23.25. Show that

$$\mu = \frac{m}{A}\sqrt{\omega^2 - (\omega_d)^2}$$

where the damping force on the plate is equal to $\mu.A.v$ ; $v$ being the velocity.

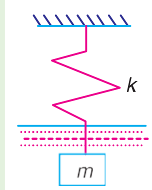


**Fig. 23.25**

The symbols $\omega$ and $\omega_d$ indicate the undamped and damped natural circular frequencies of oscillations.

13. Explain the term 'Logarithmic decrement' as applied to damped vibrations.
14. Establish an expression for the amplitude of forced vibrations.
15. Explain the term 'dynamic magnifier'.
16. What do you understand by transmissibility ?

## OBJECTIVE TYPE QUESTIONS

**1.** When there is a reduction in amplitude over every cycle of vibration, then the body is said to have

(a) free vibration (*b*) forced vibration (*c*) damped vibration

**2.** Longitudinal vibrations are said to occur when the particles of a body moves

(a) perpendicular to its axis (*b*) parallel to its axis

(c) in a circle about its axis

**3.** When a body is subjected to transverse vibrations, the stress induced in a body will be

(a) shear stress (*b*) tensile stress (*c*) compressive stress

**4.** The natural frequency (in Hz) of free longitudinal vibrations is equal to

(*a*) $\frac{1}{2\pi}\sqrt{\frac{s}{m}}$ (*b*) $\frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ (*c*) $\frac{0.4985}{\sqrt{\delta}}$

(d) any one of these

where $m$ = Mass of the body in kg,

$s$ = Stiffness of the body in N/m, and

$\delta$ = Static deflection of the body in metres.

**5.** The factor which affects the critical speed of a shaft is

(a) diameter of the disc (*b*) span of the shaft

(*c*) eccentricity (*d*) all of these

**6.** The equation of motion for a vibrating system with viscous damping is

$$\frac{d^2x}{dt^2} + \frac{c}{m} \times \frac{dx}{dt} + \frac{s}{m} \times x = 0$$

If the roots of this equation are real, then the system will be

(a) over damped (*b*) under damped (*c*) critically damped

**7.** In under damped vibrating system, if $x_1$ and $x_2$ are the successive values of the amplitude on the same side of the mean position, then the logarithmic decrement is equal to

(*a*) $x_1/x_2$ (*b*) log $(x_1/x_2)$ (*c*) $\log_e (x_1/x_2)$ (*d*) log $(x_1.x_2)$

**8.** The ratio of the maximum displacement of the forced vibration to the deflection due to the static force, is known as

(a) damping factor (*b*) damping coefficient

(*c*) logarithmic decrement (*d*) magnification factor

**9.** In vibration isolation system, if $\omega/\omega_n$ is less than $\sqrt{2}$, then for all values of the damping factor, the transmissibility will be

(a) less than unity (*b*) equal to unity (*c*) greater than unity (*d*) zero where

$\omega$ = Circular frequency of the system in rad/s, and

$\omega_n$ = Natural circular frequency of vibration of the system in rad/s.

**10.** In vibration isolation system, if $\omega/\omega_n > 1$, then the phase difference between the transmitted force and the disturbing force is

(*a*) 0° (*b*) 90° (*c*) 180° (*d*) 270°

## ANSWERS

| | | | | |
|---|---|---|---|---|
| **1.** (*c*) | **2.** (*b*) | **3.** (*b*) | **4.** (*d*) | **5.** (*d*) |
| **6.** (*a*) | **7.** (*b*) | **8.** (*d*) | **9.** (*c*) | **10.** (*c*) |

GO To FIRST

# UNIT-V
# GOVERNERS

## Introduction

The function of a governor is to regulate the mean speed of an engine, when there are variations in the load *e.g.* when the load on an engine increases, its speed decreases, therefore it becomes necessary to increase the supply of work- ing fluid. On the other hand, when the load on the engine decreases, its speed increases and thus less working fluid is required. The governor automatically controls the supply of working fluid to the engine with the varying load conditions and keeps the mean speed within certain limits.

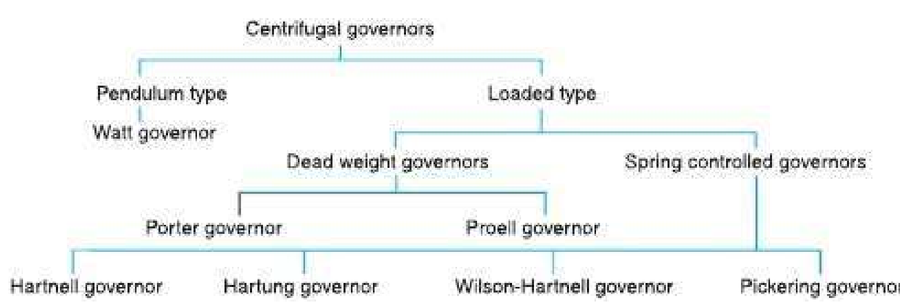
A little consideration will show, that when the load increases, the configuration of the governor changes and a valve is moved to increase the supply of the working fluid ; ***conversely,*** when the load decreases, the engine speed in- creases and the governor decreases the supply of working fluid.

**Note :** We have discussed in Chapter 16 (Art. 16.8) that the func- tion of a flywheel in an engine is entirely different from that of a governor. It controls the speed variation caused by the fluctuations of the engine turning moment during each cycle of operation. It does not control the speed variations caused by a varying load. The varying demand for power is met by the governor regulating the supply of working fluid.

## Types of Governors

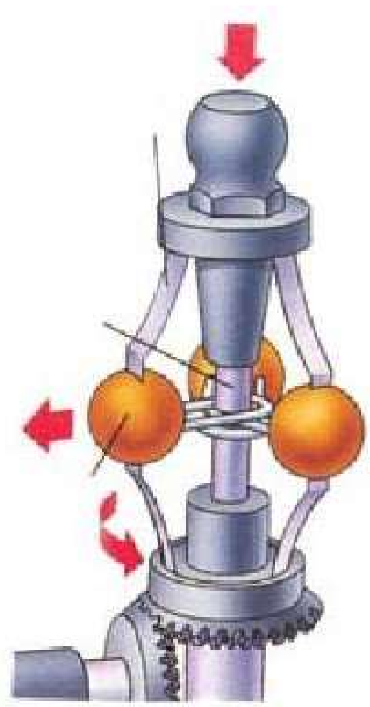
The governors may, broadly, be classified as

1. Centrifugal governors, and
2. Inertia governors.

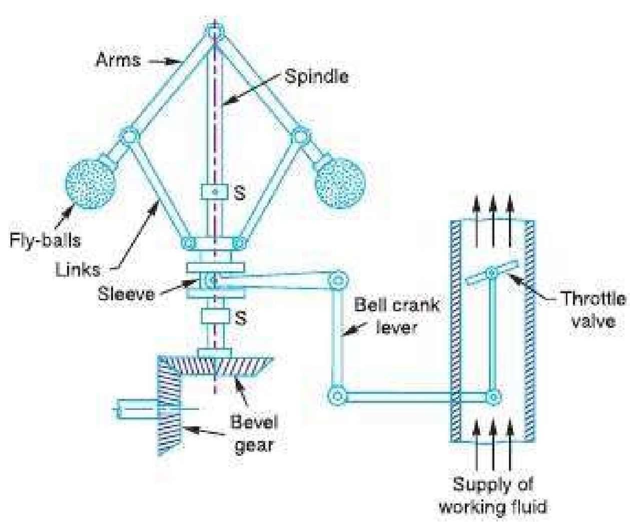



## Centrifugal Governors

The centrifugal governors are based on the balancing of centrifugal force on the rotating balls by an equal and opposite radial force, known as the ***controlling force****.It consists of two balls of equal mass, which are attached to the arms as shown in Fig. 18.1. These balls are known as ***governor balls* or *fly balls.*** The balls revolve with a spindle, which is driven by the engine through bevel gears. The upper ends of the arms are pivoted to the spindle, so that the balls may rise up or fall down as they revolve about the vertical axis. The arms are connected by the links to a sleeve, which is keyed to the spindle. This sleeve re- volves with the spindle ; but can slide up and down. The balls and the sleeve rises when the spindle speed increases, and falls when the speed decreases. In order to limit the travel of the sleeve in upward and downward directions, two stops $S$, $S$ are provided on the spindle. The sleeve is connected by a bell crank lever to a throttle valve. The supply of the working fluid de- creases when the sleeve rises and increases when it falls.



When the load on the engine increases, the engine and the governor speed decreases. This results in the decrease of centrifugal force on the balls. Hence the balls move inwards and the sleeve moves down- wards. The downward movement of the sleeve operates a throttle valve at the other end of the bell crank lever to increase the supply of working fluid and thus the engine speed is increased. In this case, the extra power output is provided to balance the increased load. When the load on the engine decreases, the engine and the governor speed increases, which results in the increase of centrifugal force on the balls. Thus the balls move outwards and the sleeve rises upwards. This upward movement of the sleeve reduces the supply of the working fluid and hence the speed is decreased. In this case, the power output is reduced.



## Terms Used in Governors

The following terms used in governors are important from the subject point of view ;

**1.** ***Height of a governor.*** It is the vertical distance from the centre of the ball to a point where the axes of the arms (or arms produced) intersect on the spindle axis. It is usually denoted by $h$.

**2.** ***Equilibrium speed.*** It is the speed at which the governor balls, arms etc., are in complete equilibrium and the sleeve does not tend to move upwards or downwards.

**3.** ***Mean equilibrium speed.*** It is the speed at the mean position of the balls or the sleeve.

**4.** ***Maximum and minimum equilibrium speeds.*** The speeds at the maximum and minimum radius of rotation of the balls, without tending to move either way are known as maximum and mini- mum equilibrium speeds respectively.

**Note :** There can be many equilibrium speeds between the mean and the maximum and the mean and the mini- mum equilibrium speeds

**5.** ***Sleeve lift.*** It is the vertical distance which the sleeve travels due to change in equilibrium speed.

## 3.4 Gravity Loaded Controlled Governors

### (a) Watt Governor

This type of governor is shown in fig-3.1 (a). It is the original form of governor as used by Watt on some of his early steam engines. In this type of governor, each ball is attached to an arm, which is pivoted on the axis of rotation. The sleeve is attached to the governor balls by arms, pin-jointed at both ends, and is free to slide along the governor shaft.

The upper arm may be suspended from the vertical spindle in three ways as shown in fig-3.3.

(i) From the axis of the spindle as shown in fig-3.3 (a).

(ii) From a point attached to a collar on the spindle so that the arm produced intersects the spindle as shown in fig-3.3 (b).

(iii) From a point to a collar so that the arm crosses the spindle as shown in fig 3.3(c).

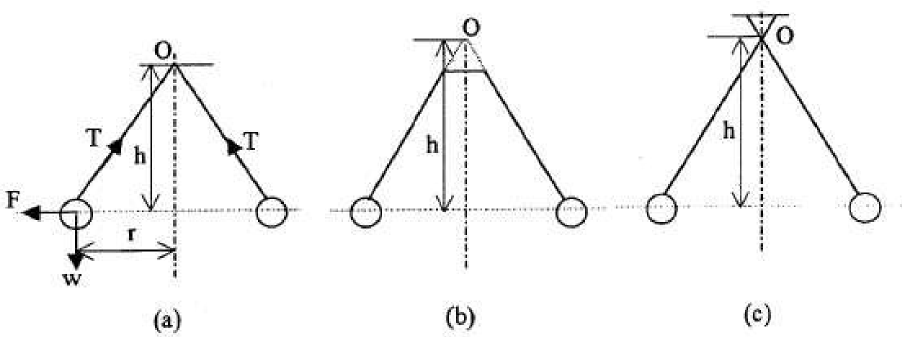



Fig-3.3

The height of the governor, which is donated by 'h' in figure, is the distance from the center of the mass to the point of intersection between the arm and the axis of the spindle.

Let 'w' be the weight of the ball, 'T' the tension in the arm and 'F' the centrifugal force when the radius to the center of the ball is 'r' and the angular velocity of the arm and the ball about the spindle axis is 'ω'.

For the simplified analysis, which follows, the weights of the sleeve, the upper ball arms, the lower links and friction are all neglected. As the weight of the lower arms and sleeve is neglected, the tensions in the lower links are negligible and hence only three forces are acting on each rotating ball.

(i) The weight 'w' acting vertically downwards

(ii) The centrifugal force '$F = \frac{w}{g}\omega^2 r$' acting radially outwards

(iii) The tension 'T' in the upper arm.

Taking moment about O, the point of intersection of the arm and the axis of the spindle, for the forces acting on the governor balls, we get

$$\frac{w}{g}\omega^2 r \times h = w \times r$$

$$h = \frac{g}{\omega^2} \qquad \text{(i)}$$

The equation (i) shows that neither the weight of the balls nor the length of the supporting arms has any influence on the height of the governor. It varies inversely as the square of the speed.

When 'g' is in cm/s$^2$ and 'ω' is in radian/s, then 'h' is in cm.

Let 'N' be the speed in rpm, then

$$\omega = \frac{2\pi N}{60} = \frac{\pi N}{30}$$

$$\therefore h = \frac{900g}{\pi^2 N^2} = \frac{900 \times 981}{\pi^2 N^2} = \frac{89560}{N^2} \text{ cm.}$$

Since the height of the governor is inversely proportional to the square of the speed it is small at high speeds and at such speeds the change in height corresponding to a small change in speed is insufficient to enable a governor of the Watt type to operate the mechanism to give the necessary change in the fuel supply or steam supply.

From the table given below it can be seen that the height diminishes very rapidly as the speed of rotation increases.

| N (rpm) | 40 | 60 | 80 | 100 | 120 | 150 | 220 |
|---|---|---|---|---|---|---|---|
| h (cm) | 55.98 | 24.88 | 13.98 | 8.96 | 6.22 | 3.98 | 2.24 |

Thus, this governor is suitable only for low speeds of rotation not exceeding 75 rpm. It might then be suggested that a speed reduction gear between engine shaft and the governor spindle would allow this governor to be used with higher speed engines. However, it should be noted that this is not a satisfactory remedy.

**(b) Porter Governor**

The type of governor, which is illustrated at fig-3.1 (b), is known as the Porter governor. The only respect in which it differs from the Watt governor is in the use of a heavily weighted sleeve. The additional downward force increases the speed of revolution required to enable the balls to rise to any pre-determined level.

Let 'w' be the weight of each ball and 'W' be the weight of the central load. $T_1$ be the tension in the upper arm and $T_2$ the tension in the suspension link. $\alpha$ and $\beta$ be the inclinations to the vertical of the upper arm and suspension links respectively. The weight of arms and weight of suspension links and the effect of friction to the movement of the sleeve are neglected.

There are several ways of determining the relation between the height 'h' and the speed '$\omega$'. In this chapter, two methods are used to derive the relation.

**(i) Instantaneous Center Method**

Consider the equilibrium of the forces acting on the suspension link 'AC', which is shown in fig-3.4. These forces are 'F, w and $T_1$ at C and $\frac{W}{2}$ and Q at A. The equation connecting 'F, w and W' is derived by taking moment about I, the point of intersection of the lines of action of forces $T_1$ and Q. This point of intersection I is also the instantaneous center of the link AC. The point I lies at the point of intersection of BC produce and a line drawn through A perpendicular to the axis of the governor spindle.

Taking moment about I,

$$F \times CD = w \times ID + \frac{W}{2}(ID + DA)$$

$$F = w \times \frac{ID}{CD} + \frac{W}{2}\left(\frac{ID}{CD} + \frac{DA}{CD}\right)$$

$$= w \tan\alpha + \frac{W}{2}(\tan\alpha + \tan\beta)$$

$$= \left\{\frac{W}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right) + w\right\}\tan\alpha$$

$$= \left\{\frac{W}{2}(1 + k) + w\right\}\tan\alpha$$

where $k = \frac{\tan\beta}{\tan\alpha}$.

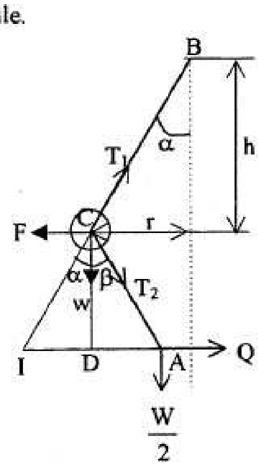



Fig-3.4

If 'h' be the height of the governor, then $\tan\alpha = \frac{r}{h}$. Further, we have $F = \frac{w}{g}\omega^2 r$.

Therefore, we get

$$\frac{w}{g}\omega^2 r = \left[\frac{W}{2}(1+k)+w\right]\frac{r}{h} \qquad \text{(or)}$$

$$\omega^2 = \left[\frac{\frac{W}{2}(1+k)+w}{w}\right]\frac{g}{h} \qquad \text{(i)}$$

When the length of the arms and the suspension links are of equal length and the axis of the joints at B and A either intersect the governor spindle or are at equal distances from the governor spindle the value 'k' is equal to 1 and the equation (i) reduces to the form

$$\omega^2 = \left(\frac{W+w}{w}\right)\frac{g}{h} \qquad \text{(ii)}$$

When the lengths of the arms are unequal and the axes of the joints at B and A are at different distances from the governor spindle the k will have a different value for each radius of rotation of the governor balls, This value of 'k' can be best found by calculating the value of $\alpha$ and $\beta$. It should be noted that when 'k' is not equal to 1, its value changes as the height of the governor changes.

For the simple Watt governor, the weight of the sleeve W is negligible and we have either from equation (i) or (ii) the relation $\omega^2 = \frac{g}{h}$ which has derived earlier.

**(ii) Equilibrium Method**

The governor sleeve, which is loaded by the weight W is in equilibrium under a system of three forces, W the load on the sleeve and the tensions $T_2$ in the two lowered suspension links. As the system of forces is in equilibrium, the force triangle drawn for these forces must be a closed one as shown in fig-3.5 (a).

The pin joint C between the upper arm and the lower suspension link must be in equilibrium under the action of the four forces as under:

(i) The weight of the ball 'w'

(ii) Radially outwards acting centrifugal force $F = \frac{w}{g}\omega^2 r$

(iii) Tension $T_1$ in the upper arm

(iv) Tension $T_2$ in the lower suspension link.

These four forces must form a closed polygon as shown in fig-3.5 (b).

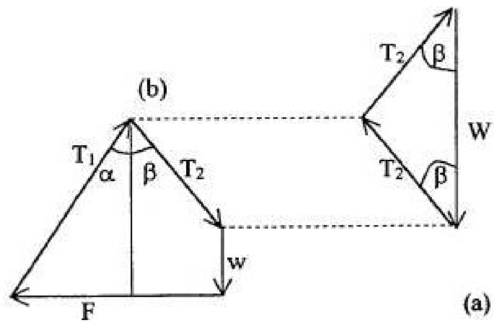



Fig-3.5

From force triangle for the sleeve, we get

$$W = 2\,T_2 \cos\beta \qquad \text{(or)} \qquad T_2 = \frac{W}{2\cos\beta} \qquad \text{(iii)}$$

From the polygon of forces on the ball, we have

$$T_1 \cos\alpha = T_2 \cos\beta + w \qquad \text{(resolving vertically)} \qquad \text{(iv)}$$

Resolving horizontally,

$$F = T_1 \sin\alpha + T_2 \sin\beta \qquad \text{(v)}$$

From equation (iv)

$$T_1 = \frac{\frac{W}{2} + w}{\cos\alpha}$$

When the value of $T_1$ and $T_2$ are substituted in the equation (v),

$$F = \left(\frac{W}{2} + w\right)\tan\alpha + \frac{W}{2}\tan\beta$$

$$= \left[\frac{W}{2}(1+k) + w\right]\tan\alpha \qquad \text{(v)}$$

where $k = \frac{\tan\beta}{\tan\alpha}$.

By substituting the value of $\tan\alpha$ and F, equation (i), which is derived earlier, can be done.

**(c) Proell Governor**

Fig-3.1(c) shows a type of Proell governor. This governor is similar to the Porter governor except that the revolving balls are attached to the extensions of the lower links. This has the effect of reducing the change of speed necessary for a given sleeve movement. In other words the governor is made more sensitive.

The action of this governor is again similar to that of the other governors described earlier. The analysis of the Proell governor can be done by considering the equilibrium of the lower arm, which is referred fig-3.8.

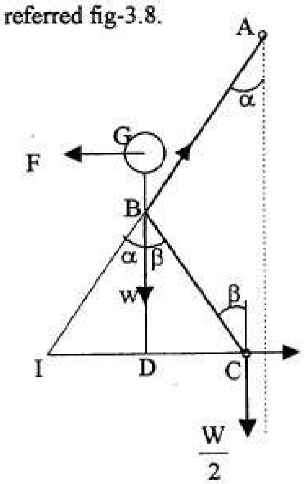



Fig-3.8

There are five forces acting on the lower link:

(i) The centrifugal force F, acting radially outwards, through the center of the gravity of the ball

(ii) The weight 'w', acting vertically downwards through the center of gravity of the ball

(iii) The pull $\frac{W}{2}$ at C acting vertically downwards

(iv) The tension $T_1$ along the length of the link AB

(v) Reaction at C along a line at right angles to the axis of the governor spindle.

The instantaneous center of the lower suspension link BC lies at the point of intersection of AB produced and a line drawn through C perpendicular to the axis of the governor spindle. It is assumed that the extension BG of the lower suspension link BC is vertical for the given configuration.

Take moment about I, the instantaneous center of the lower suspension link. The tension $T_1$ and the reaction at C give no moment. Therefore,

$$F \times DG = w \times ID + \frac{W}{2} \times (ID + DC) \qquad \text{(i)}$$

Dividing both sides by BD,

$$F \times \frac{DG}{BD} = w \times \frac{ID}{BD} + \frac{W}{2}\left[\frac{ID}{BD} + \frac{DC}{BD}\right]$$

$$= w \tan\alpha + \frac{W}{2}[\tan\alpha + \tan\beta]$$

$$= \left[w + \frac{W}{2}\right]\tan\alpha + \frac{W}{2}\tan\beta$$

$$\therefore F = \frac{BD}{DG}\left\{\left[w + \frac{W}{2}\right]\tan\alpha + \frac{W}{2}\tan\beta\right\} \qquad \text{(ii)}$$

Let $\frac{\tan\beta}{\tan\alpha} = k$

$$\therefore F = \frac{BD}{DG}\left\{\frac{W}{2}(1+k) + w\right\}\tan\alpha \qquad \text{(iii)}$$

But, $\tan\alpha = \frac{r}{h}$ and $F = \frac{w}{g}\omega^2 r$ (iv)

Substituting the values given by equation (iv) in equation (iii),

$$\frac{w}{g}\omega^2 r = \frac{BD}{DG}\left\{\frac{W}{2}(1+k) + w\right\}\frac{r}{h}$$

$$\omega^2 = \frac{g}{h} \times \frac{BD}{DG}\left\{\frac{\frac{W}{2}(1+k) + w}{w}\right\} \qquad \text{(v)}$$

Thus, the effect of placing the ball at G, instead of at the pin joint B is to reduce the equilibrium speed for given values of the height of the governor, the weight of the ball and the weight of the sleeve. Hence in order to give the same equilibrium speed for the given height and the weight of the sleeve, the smaller ball is required in Proell governor than that in Porter governor.

## 3.5 Spring Loaded Controlled Governors

In spring loaded controlled governors the control of speed is affected either wholly or in part by means of springs. Some of the representative of spring loaded controlled governors are shown in fig-3.2.

The spring loaded controlled governors posses the following advantages over the gravity loaded controlled governors.

(i) The spring loaded controlled governors may be operated at very high speeds.

(ii) With proper proportioning the spring loaded controlled governors can be made both powerful and capable of very closed regulation.

(iii) It can be much smaller in over all size.

(iv) As it does not depend on gravity for its action, it may revolve about a horizontal, vertical or inclined axis.

In spring loaded controlled governors the spring may be placed upon the axis of rotation or they may be transverse as shown in fig-3.2.

### (a) Spring loaded Controlled Governor of the Hartnell Type

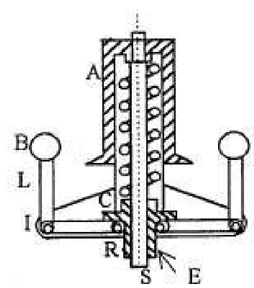



Fig-3.10

Fig-3.10 shows spring loaded controlled governor of Hartnell type. Two bell crank levers L are mounted on pins I, carried by the frame A, which is attached to the rotating spindle S. Each lever carries a ball B at the end of one arm and a roller R the end of the other. The centrifugal forces of the balls cause the rollers R to press against the collar C on the sleeve E. The upward pressure of the rollers on the collar of the sleeve is balanced by the downward thrust of the helical spring, which is in compression. The angle of the bell crank lever is usually 90° but in practice it may be grater.

Let w be the weight of each ball, S the spring force exerted on the sleeve, k the stiffness of the spring, ω the speed of rotation, r the radius of rotation, a and b the lengths of the vertical and horizontal arms of the bell crank lever and F the centrifugal force on the ball.

By taking moment about the fulcrum of the lever, neglecting the effect of pull of gravity on the governor balls and arms,

$$F \times a = \frac{S}{2} \times b$$

or $$S = 2F\frac{a}{b} \quad \text{(i)}$$

It is assumed that the arms are mutually perpendicular and the lines of action of forces are at right angles to the arm.

Let the suffixes 1 and 2 denote the values of maximum and minimum radii respectively. Then at maximum radius

$$S_1 = 2F_1\frac{a}{b} \quad \text{(ii)}$$

At minimum radius, $$S_2 = 2F_2\frac{a}{b} \quad \text{(iii)}$$

$$\therefore S_1 - S_2 = 2\frac{a}{b}(F_1 - F_2)$$

Let θ be the angular movement of the bell crank lever from the position of minimum radius to the position of the maximum radius, then

$$(r_1 - r_2) = a\theta \quad \text{(iv)}$$

If h be the lift of the sleeve, then

$$h = b\theta \quad \text{(v)}$$

Dividing equation (v) by (iv),

$$\frac{h}{r_r - r_2} = \frac{b}{a} \quad \text{(or)}$$

$$h = \frac{b}{a}(r_1 - r_2) \quad \text{(vi)}$$

The difference in the forces exerted by the compressed spring in the two positions is $S_1 - S_2$; therefore, the force per unit compression is known as the stiffness of the spring. The stiffness of the spring is denoted by k.